

Harnack inequalities and W -entropy formula for Witten Laplacian on Riemannian manifolds with K -super Perelman Ricci flow

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February 9, 2016

Abstract. In this paper, we prove logarithmic Sobolev inequalities and derive the Hamilton Harnack inequality for the heat semigroup of the Witten Laplacian on complete Riemannian manifolds equipped with K -super Perelman Ricci flow. We establish the W -entropy formula for the heat equation of the Witten Laplacian and prove a rigidity theorem on complete Riemannian manifolds satisfying the $CD(K, m)$ condition, and extend the W -entropy formula to time dependent Witten Laplacian on compact Riemannian manifolds with (K, m) -super Perelman Ricci flow, where $K \in \mathbb{R}$ and $m \in [n, \infty]$ are two constants. Finally, we prove the Li-Yau and the Li-Yau-Hamilton Harnack inequalities for positive solutions to the heat equation $\partial_t u = Lu$ associated to the time dependent Witten Laplacian on compact or complete manifolds equipped with variants of the (K, m) -super Ricci flow.

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*Research partially supported by the China Scholarship Council.

[†]Research supported by NSFC No. 11371351, Key Laboratory RCSDS, CAS, No. 2008DP173182, and a Hundred Talents Project of AMSS, CAS.

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1 Introduction

1.1 The differential Harnack inequality

Differential Harnack inequality is an important tool in the study of heat equations and geometric flows on Riemannian manifolds. Let M be an n dimensional complete Riemannian manifold, u be a positive solution to the heat equation

$$\partial_t u = \Delta u. \quad (1)$$

In their famous paper [12], Li and Yau proved that, if $Ric \geq -K$, where $K \geq 0$ is a positive constant, then for all $\alpha > 1$,

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{\partial_t u}{u} \leq \frac{n\alpha^2}{2t} + \frac{n\alpha^2 K}{\sqrt{2}(\alpha - 1)}. \quad (2)$$

In particular, if $Ric \geq 0$, then taking $\alpha \rightarrow 1$, the Li-Yau differential Harnack inequality [12] holds

$$\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{n}{2t}. \quad (3)$$

In [10], Hamilton proved a dimension free Harnack inequality on compact Riemannian manifolds with Ricci curvature bounded from below. More precisely, if

$$Ric \geq -K,$$

then, for any positive and bounded solution u to the heat equation (1), it holds

$$\frac{|\nabla u|^2}{u^2} \leq \left(\frac{1}{t} + 2K \right) \log(A/u), \quad \forall x \in M, t > 0, \quad (4)$$

where

$$A := \sup\{u(t, x) : x \in M, t \geq 0\}.$$

Indeed, the same result holds on complete Riemannian manifolds with Ricci curvature bounded from below. Under the same condition $Ric \geq -K$, Hamilton also proved the following Li-Yau type Harnack inequality for any positive solution to the heat equation (1)

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt} \frac{\partial_t u}{u} \leq \frac{n}{2t} e^{4Kt}. \quad (5)$$

In particular, when $K = 0$, the above inequality reduces to the Li-Yau Harnack inequality (3) on complete Riemannian manifolds with non-negative Ricci curvature.

In this paper, we refer the inequality (4) Hamilton's (differential) Harnack inequality, and refer the inequalities (3), (2) and (5) the Li-Yau-Hamilton (differential) Harnack inequality.

1.2 The W -entropy formula

Let M be a closed manifold. In [23], Perelman introduced the \mathcal{F} -entropy on the space of Riemannian metrics and smooth functions as follows

$$\mathcal{F}(g, f) = \int_M (R + |\nabla f|^2) e^{-f} dv,$$

where $g \in \mathcal{M} = \{g : \text{Riemannian metric on } M\}$, $f \in C^\infty(M)$, R denotes the scalar curvature on (M, g) , and dv denotes the volume measure. Under the constraint condition that

$$dm = e^{-f} dv$$

is fixed, Perelman [23] proved that the gradient flow of \mathcal{F} with respect to the standard L^2 -metric on $\mathcal{M} \times C^\infty(M)$ is given by the following modified Ricci flow for g together with the conjugate heat equation for f , i.e.,

$$\begin{aligned} \partial_t g &= -2(Ric + \nabla^2 f), \\ \partial_t f &= -\Delta f - R. \end{aligned}$$

Moreover, Perelman [23] introduced the remarkable W -entropy as follows

$$W(g, f, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n] \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv, \quad (6)$$

where $\tau > 0$, and $f \in C^\infty(M)$ satisfies the following condition

$$\int_M (4\pi\tau)^{-n/2} e^{-f} dv = 1.$$

By [23], it is known that, if $(g(t), f(t), \tau(t))$ satisfies the evolution equations

$$\partial_t g = -2Ric, \quad \partial_t f = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}, \quad \partial_t \tau = -1, \quad (7)$$

where the first one is Hamilton's Ricci flow, and the second one is the corresponding conjugate heat equation, then the following Perelman entropy formula holds

$$\frac{d}{dt} W(g, f, \tau) = 2 \int_M \tau \left| Ric + \nabla^2 f - \frac{g}{2\tau} \right|^2 \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv. \quad (8)$$

This implies that the W -entropy is increasing in τ and the monotonicity is strict except that M is a shrinking Ricci soliton

$$Ric + \nabla^2 f = \frac{g}{2\tau}.$$

As an application of the above entropy formula, Perelman [23] derived the non local collapsing theorem for the Ricci flow, which plays an important rôle for ruling out cigars, the one part of the singularity classification for the final resolution of the Poincaré conjecture and geometrization conjecture.

Since Perelman's preprint [23] was published on Arxiv in 2002, many people have studied the W -like entropy for other geometric flows on Riemannian manifolds [21, 22, 7, 19, 11]. In [21, 22], Ni studied the W -entropy for the linear heat equation on complete Riemannian manifolds. More precisely, let (M, g) be an n -dimensional complete Riemannian manifold, let

$$u = \frac{e^{-f}}{(4\pi t)^{n/2}}$$

be a positive solution to the linear heat equation

$$\partial_t u = \Delta u \quad (9)$$

with $\int_M u(x, 0) dv(x) = 1$. The W -entropy for the linear heat equation (9) is defined by

$$W(f, t) = \int_M [t|\nabla f|^2 + f - n] \frac{e^{-f}}{(4\pi t)^{n/2}} dv. \quad (10)$$

In [21], Ni proved the following entropy formula

$$\frac{dW(f, t)}{dt} = -2 \int_M t \left(\left| \nabla^2 f - \frac{g}{2t} \right|^2 + Ric(\nabla f, \nabla f) \right) \frac{e^{-f}}{(4\pi t)^{n/2}} dv. \quad (11)$$

This yields that the W -entropy for the linear heat equation (9) is decreasing on complete Riemannian manifolds with non-negative Ricci curvature.

In [15, 17], the second author of this paper introduced the W -entropy for the heat equation associated with the Witten Laplacian and proved the monotonicity and rigidity results on complete Riemannian manifolds with non-negative m -dimensional Bakry-Emery Ricci curvature condition. More precisely, let (M, g) be a complete Riemannian manifold, $\phi \in C^2(M)$. Let

$$L = \Delta - \nabla \phi \cdot \nabla$$

be the Witten Laplacian on (M, g) with respect to μ , where

$$d\mu = e^{-\phi} dv.$$

For any $m \in [n, \infty)$, let

$$Ric_{m,n}(L) = Ric + \nabla^2 \phi - \frac{\nabla \phi \otimes \nabla \phi}{m - n},$$

be the m -dimensional Bakry-Emery Ricci curvature of L , where $m = n$ if and only if ϕ is identically constant. Let $u = \frac{e^{-f}}{(4\pi t)^{m/2}}$ be the fundamental solution to the heat equation

$$\partial_t u = Lu.$$

Let

$$H_m(u, t) = - \int_M u \log u d\mu - \frac{m}{2} (1 + \log(4\pi t)). \quad (12)$$

Define the W -entropy by the Boltzmann formula

$$W_m(u, t) = \frac{d}{dt} (t H_m(u, t)). \quad (13)$$

Then

$$\frac{d}{dt} H_m(u, t) = - \int_M \left(L \log u + \frac{m}{2t} \right) u d\mu, \quad (14)$$

and

$$W_m(u, t) = \int_M [t|\nabla \log u|^2 + f - m] \frac{e^{-f}}{(4\pi t)^{m/2}} d\mu. \quad (15)$$

Moreover, under the condition that (M, g) is a complete Riemannian manifold with bounded geometry condition, and $\phi \in C^4(M)$ with $\nabla\phi \in C_b^3(M)$, we have¹

$$\begin{aligned} \frac{dW_m(u, t)}{dt} = & -2 \int_M t \left(\left| \nabla^2 f - \frac{g}{2t} \right|^2 + Ric_{m,n}(L)(\nabla f, \nabla f) \right) u d\mu \\ & - \frac{2}{m-n} \int_M t \left(\nabla\phi \cdot \nabla f + \frac{m-n}{2t} \right)^2 u d\mu. \end{aligned} \quad (16)$$

In particular, if (M, g, ϕ) satisfies the bounded geometry condition and $Ric_{m,n}(L) \geq 0$, then the W -entropy is decreasing in time t , i.e.,

$$\frac{dW_m(u, t)}{dt} \leq 0, \quad \forall t \geq 0.$$

Moreover, $\frac{dW_m(u, t)}{dt} = 0$ holds at some $t = t_0 > 0$ if and only if M is isometric to \mathbb{R}^n , $m = n$,

$\phi = C$ for some constant $C \in \mathbb{R}$, and $u(x, t) = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}$ for all $x \in \mathbb{R}^n$ and $t > 0$.

Here we say that (M, g) satisfies the bounded geometry condition if the Riemannian curvature tensor $Riem$ and its covariant derivatives $\nabla^k Riem$ are uniformly bounded on M , $k = 1, 2, 3$. The bounded geometry condition and the assumption $\phi \in C^4(M)$ with $\nabla\phi \in C_b^3(M)$ are only required in order to allow us to exchange the time derivatives and the integration of $u \log u$ on complete non-compact Riemannian manifolds.

In [18], when $m \in \mathbb{N}$, we gave a direct proof of the W -entropy formula (16) for the Witten Laplacian by applying Ni's W -entropy formula (10) for the usual Laplacian to $M \times S^{m-n}$ equipped with a suitable warped product Riemannian metric, and gave a natural geometric interpretation for the third term in the W -entropy formula (16) for the Witten Laplacian. We have further proved the W -entropy formula for time dependent Witten Laplacian on Riemannian manifolds with time dependent metrics and potentials. In particular, if $d\mu = e^{-\phi} dv$ is fixed and if

$$\frac{1}{2} \frac{\partial g}{\partial t} + Ric_{m,n}(L) \geq 0,$$

then the W -entropy for the time dependent Witten Laplacian $L = \Delta_{g(t)} - \nabla_{g(t)}\phi(t) \cdot \nabla_{g(t)}$ is decreasing in time. For details, see [18].

1.3 The purpose of our paper

In [13, 2, 15], the Li-Yau Harnack inequality (3) has been extended to positive solutions of the heat equation associated to the Witten Laplacian on complete Riemannian manifolds with non-negative m -dimensional Bakry-Emery Ricci curvature. More precisely, if $Ric_{m,n}(L) \geq 0$, or equivalently, if the $CD(0, m)$ condition holds on (M, g, ϕ) , then

$$L \log u + \frac{m}{2t} \geq 0. \quad (17)$$

In [17], the Hamilton Harnack inequality has been also extended to positive and bounded solutions to the heat equation of the Witten Laplacian on complete Riemannian manifolds with the infinite dimensional Bakry-Emery Ricci curvature bounded from below, or equivalently, on (M, g, ϕ) with the $CD(-K, \infty)$ condition, i.e., $Ric(L) = Ric + \nabla^2\phi \geq -K$, where $K \in \mathbb{R}$ is a constant. As already mentioned here, the main ingredient in these work is the introduction of the notion of finite or infinite dimensional Bakry-Emery Ricci curvature,

¹In 2006, the second author of this paper proved the W -entropy formula (16) for all positive solutions to the heat equation $\partial_t u = Lu$ on compact Riemannian manifolds with fixed metrics and potentials. See [14].

which plays an important role to substitute that of Ricci curvature in the study of geometric analysis for the usual Laplace-Beltrami operator on Riemannian manifolds.

On the other hand, from (12), (13) and (14), we can see that there exists an essential and deep relationship among the W -entropy for the Witten Laplacian, the Gaussian heat kernel on \mathbb{R}^m and the Li-Yau Harnack inequality (17) on complete Riemannian manifolds satisfying the $CD(0, m)$ condition. Indeed, the W -entropy for the heat equation $\partial_t u = Lu$ is defined by the Boltzmann formula (13) together with (12), in which the quantity $H_m(u, t) = -\int_M u \log u d\mu - \frac{m}{2}(\log(4\pi t) + 1)$ is defined as the difference between the Boltzmann-Shannon entropy $\bar{H}(u) = -\int_M u \log u d\mu$ for the heat kernel measure $u(x, t)d\mu(x)$ of the Witten Laplacian on M and the Boltzmann-Shannon entropy $H(\bar{u}) = -\int_{\mathbb{R}^m} \bar{u} \log \bar{u} dx = \frac{m}{2}(\log(4\pi t) + 1)$ of the Gaussian heat kernel measure $\bar{u}(x, t)dx$ on \mathbb{R}^m with $\bar{u}(x, t) = \frac{e^{-\frac{\|x\|^2}{4t}}}{(4\pi t)^{m/2}}$. By (14), the time derivative of $H_m(u, t)$ is given by the integral of the Li-Yau Harnack quantity $L \log u + \frac{m}{2t}$ with respect to the heat kernel measure $u(x, t)d\mu(x)$. The W -entropy formula (16) does not only imply the monotonicity of the W -entropy on complete Riemannian manifolds with the $CD(0, m)$ condition, but also allows us to prove a rigidity theorem which characterizes the unique equilibrium state of the W -entropy on the canonical ensemble of all complete Riemannian manifolds satisfying the $CD(0, m)$ condition. In [18], we defined the W -entropy for the heat equation of the time dependent Witten Laplacian by the same formulas (12) and (13), and proved that the W -entropy introduced in this way is monotonically decreasing in time on compact Riemannian manifolds with $\frac{1}{2} \frac{\partial g}{\partial t} + Ric_{m,n}(L) \geq 0$.

Now it is very natural to raise the following problems: (1) How to define the W -entropy functional for the heat equation associated with the Witten Laplacian on complete Riemannian manifolds satisfying the $CD(K, \infty)$ condition or the $CD(K, m)$ condition for $K \in \mathbb{R}$ and $m \in [n, \infty)$? (2) Can we establish the monotonicity and rigidity theorems for the W -entropy associated with the Witten Laplacian on complete Riemannian manifolds satisfying general curvature-dimension condition? (3) What happens on Riemannian manifolds with time dependent metrics and potentials? (4) How to extend the Li-Yau and the Li-Yau-Hamilton Harnack inequalities to positive solutions to the heat equation associated to the time dependent Witten Laplacian on compact or complete manifolds with time dependent metrics and potentials? Indeed, we have been asked these questions for many times by many people during the past years.

The purpose of this paper is to study these problems. We first prove the logarithmic Sobolev inequality and the reversal logarithmic Sobolev inequality and derive the Hamilton Harnack inequality for the heat equation of the Witten Laplacian on complete Riemannian manifolds with time dependent metrics and potentials evolving along K -super Perelman Ricci flow. Then we introduce the W -entropy and prove the W -entropy formula and rigidity theorem for the Witten Laplacian on complete Riemannian manifolds with fixed metrics and potentials satisfying the $CD(K, m)$ condition, for $K \in \mathbb{R}$ and $m \in [n, \infty]$. Moreover, we extend the W -entropy results to compact Riemannian manifolds with time dependent metrics and potentials evolving along K -super Perelman Ricci flow. Finally, we prove the Li-Yau and Li-Yau-Hamilton Harnack inequalities for positive solutions to the heat equation associated to the time dependent Witten Laplacian on compact or complete manifolds with (K, m) -super Ricci flow.

1.4 Statement of main results

To state our results, let us first introduce some notations. Let M be a complete Riemannian manifold with a fixed Riemannian metric g , $\phi \in C^2(M)$ and $d\mu = e^{-\phi} dv$, where v is the Riemannian volume measure on (M, g) . The Witten Laplacian on (M, g) with respect to

the weighted volume measure μ or the potential function ϕ is defined by

$$L = \Delta - \nabla\phi \cdot \nabla.$$

For all $u, v \in C_0^\infty(M)$, the following integration by parts formula holds

$$\int_M \langle \nabla u, \nabla v \rangle d\mu = - \int_M Luv d\mu = - \int_M uLv d\mu.$$

In [1], Bakry and Emery proved that for all $u \in C_0^\infty(M)$,

$$L|\nabla u|^2 - 2\langle \nabla u, \nabla Lu \rangle = 2|\nabla^2 u|^2 + 2Ric(L)(\nabla u, \nabla u), \quad (18)$$

where

$$Ric(L) = Ric + \nabla^2\phi.$$

The formula (18) can be viewed as a natural extension of the Bochner-Weitzenböck formula. The quantity $Ric(L) = Ric + \nabla^2\phi$, called the infinite dimensional Bakry-Emery Ricci curvature on the weighted Riemannian manifolds (M, g, ϕ) . It plays as a good substitute of the Ricci curvature in many problems in comparison geometry and analysis on complete Riemannian manifolds with smooth weighted volume measures. See [1, 2, 8, 9, 13, 15, 20, 29] and reference therein.

Following [1, 20, 13], we introduce the m -dimensional Bakry-Emery Ricci curvature on (M, g, ϕ) by

$$Ric_{m,n}(L) := Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n},$$

where $m \geq n$ is a constant, and $m = n$ if and only if ϕ is a constant. When $m = \infty$, we have $Ric_{\infty,n}(L) = Ric(L)$. Following [2], we say that the Witten Laplacian L satisfies the $CD(K, \infty)$ condition if $Ric(L) \geq K$, and L satisfies the $CD(K, m)$ condition if $Ric_{m,n}(L) \geq K$. Recall that, when $m \in \mathbb{N}$, the m -dimensional Bakry-Emery Ricci curvature $Ric_{m,n}(L)$ has a very natural geometric interpretation. Indeed, consider the warped product metric on $M^n \times S^{m-n}$ defined by

$$\tilde{g} = g_M \oplus e^{-\frac{2\phi}{m-n}} g_{S^{m-n}}.$$

where S^{m-n} is the unit sphere in \mathbb{R}^{m-n+1} with the standard metric $g_{S^{m-n}}$. By a classical result in Riemannian geometry, the quantity $Ric_{m,n}(L)$ is equal to the Ricci curvature of the above warped product metric \tilde{g} on $M^n \times S^{m-n}$ along the horizontal vector fields. See [20, 13, 29].

Let $(M, g(t), \phi(t), t \in [0, T])$ be a complete Riemannian manifold equipped with a family of time dependent Riemannian metrics $g(t)$ and potential functions $\phi(t)$, $t \in [0, T]$. In this paper, we call $(M, g(t), \phi(t), t \in [0, T])$ a (K, m) -super Ricci flow if the metric $g(t)$ and the potential function $\phi(t)$ satisfy the following inequality

$$\frac{1}{2} \frac{\partial g}{\partial t} + Ric_{m,n}(L) \geq Kg.$$

When $m = \infty$, i.e., if the metric $g(t)$ and the potential function $\phi(t)$ satisfy the following inequality

$$\frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) \geq Kg,$$

we call $(M, g(t), \phi(t), t \in [0, T])$ a K -super Perelman Ricci flow. As mentioned in Section 1.2 and changing f by ϕ , the modified Ricci flow $\frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) = 0$ was introduced by

Perelman [23] and can be regarded as the gradient flow of $\mathcal{F}(g, \phi) = \int_M (R + |\nabla \phi|^2) e^{-\phi} dv$ on $\mathcal{M} \times C^\infty(M)$ under the constraint condition that $dm = e^{-\phi} dv$ does not change in time.

We now state the main results and describe the organisation of this paper.

In Section 2, we prove the following result which describes the equivalence between the K -super Perelman Ricci flow property and the logarithmic Sobolev inequality, the reversal logarithmic Sobolev inequality for the heat semigroup associated with the time dependent Witten Laplacian on manifolds with time dependent metrics and potentials.

Theorem 1.1 *Let M be a complete Riemannian manifold equipped with a family of time dependent metrics and C^2 -potentials $(g(t), \phi(t), t \in [0, T])$. Let $L = \Delta_{g(t)} - \nabla_{g(t)} \phi(t) \cdot \nabla_{g(t)}$ be the time dependent weighted Laplacian on $(M, g(t), \phi(t))$, $P_{s,t} f = u(\cdot, t)$ be a positive solution to the heat equation $\partial_t u = Lu$ with the initial condition $u(\cdot, s) = f$, where $0 \leq s < t \leq T$, and f is a C^1 smooth and positive function on M . Then $(g(t), \phi(t), t \in [0, T])$ satisfies a K -super Perelman Ricci flow equation*

$$\frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) \geq -K, \quad (19)$$

where $K \geq 0$ is a constant, if and only if for $0 \leq s < t \leq T$, the following logarithmic Sobolev inequality holds

$$P_{s,t}(f \log f) - P_{s,t} f \log P_{s,t} f \leq \frac{e^{2K(t-s)} - 1}{2K} P_{s,t} \left(\frac{|\nabla f|^2}{f} \right),$$

or the reversal logarithmic Sobolev inequality holds

$$\frac{|\nabla P_{s,t} f|^2}{P_{s,t} f} \leq \frac{2K}{1 - e^{-2K(t-s)}} (P_{s,t}(f \log f) - P_{s,t} f \log P_{s,t} f). \quad (20)$$

As a corollary of the above theorem, we derive the following Hamilton Harnack inequality for the positive solution of the heat equation $\partial_t u = Lu$ on complete Riemannian manifolds with K -super Perelman Ricci flow.

Theorem 1.2 *Let M be a complete Riemannian manifold equipped with a family of time dependent metrics and C^2 -potentials $(g(t), \phi(t), t \in [0, T])$ satisfying a K -super Perelman Ricci flow equation*

$$\frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) \geq -K.$$

where $K \geq 0$ is a constant independent of $t \in [0, T]$. Let u be a positive and bounded solution to the heat equation

$$\partial_t u = Lu,$$

where

$$L = \Delta_{g(t)} - \nabla_{g(t)} \phi(t) \cdot \nabla_{g(t)}$$

is the time dependent Witten Laplacian on $(M, g(t), \phi(t))$. Then for all $x \in M$ and $t > 0$,

$$\frac{|\nabla u|^2}{u^2} \leq \frac{2K}{1 - e^{-2Kt}} \log(A/u), \quad (21)$$

where

$$A := \sup\{u(t, x) : x \in M, t \geq 0\}.$$

In particular, the Hamilton Harnack inequality holds

$$\frac{|\nabla u|^2}{u^2} \leq \left(\frac{1}{t} + 2K \right) \log(A/u). \quad (22)$$

In the case $K = 0$, i.e., $(M, g(t), \phi(t), t \in [0, T])$ is a complete Riemannian manifold equipped with the super Perelman Ricci flow

$$\frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) \geq 0,$$

we have

$$\frac{|\nabla u|^2}{u^2} \leq \frac{1}{t} \log \frac{A}{u}.$$

In particular, taking $\phi = 0$, $m = n$ and $L = \Delta$, we have the Hamilton Harnack inequality on complete Riemannian manifolds with K -super Ricci flow.

Theorem 1.3 *Let M be a complete Riemannian manifold equipped with a family of Riemannian metrics $(g(t), t \in [0, T])$ evolving along a K -super Ricci flow*

$$\frac{1}{2} \frac{\partial g}{\partial t} + Ric \geq -K,$$

where $K \geq 0$ be a constant independent of $t \in [0, T]$. Let u be a positive and bounded solution to the heat equation

$$\partial_t u = \Delta u.$$

Let $A := \sup\{u(t, x) : x \in M, t \geq 0\}$. Then for all $x \in M$ and $t > 0$,

$$\frac{|\nabla u|^2}{u^2} \leq \frac{2K}{1 - e^{-2Kt}} \log(A/u).$$

In particular, the Hamilton Harnack inequality holds

$$\frac{|\nabla u|^2}{u^2} \leq \left(\frac{1}{t} + 2K \right) \log(A/u).$$

In the case $K = 0$, i.e., $(M, g(t), t \in [0, T])$ is a complete Riemannian manifold equipped with a super Ricci flow

$$\frac{1}{2} \frac{\partial g}{\partial t} + Ric \geq 0,$$

we have ²

$$\frac{|\nabla u|^2}{u^2} \leq \frac{1}{t} \log \frac{A}{u}. \quad (23)$$

Moreover, we also prove the following theorem which extends the Li-Yau-Hamilton Harnack inequality (5) to positive solutions of the heat equation $\partial_t u = Lu$ on complete Riemannian manifolds with fixed metrics and potentials satisfying the $CD(-K, m)$ condition.

²In [28], Qi S. Zhang proved (23) for the heat equation $\partial_t u = \Delta u$ on compact or complete Riemannian manifolds equipped with the Ricci flow $\partial_t g = -2Ric$.

Theorem 1.4 *Let (M, g) be a complete Riemannian manifold with a C^2 -potential ϕ . Suppose that there exist some constants $m \geq n$ and $K \geq 0$ such that*

$$\text{Ric}_{m,n}(L) \geq -K.$$

Let u be a positive solution of the heat equation

$$\partial_t u = Lu.$$

Then the Li-Yau-Hamilton Harnack inequality holds

$$\frac{\partial_t u}{u} - e^{-2Kt} \frac{|\nabla u|^2}{u^2} + e^{2Kt} \frac{m}{2t} \geq 0.$$

In particular, if $K = 0$, i.e., $\text{Ric}_{m,n}(L) \geq 0$, then the Li-Yau Harnack inequality holds

$$\frac{\partial_t u}{u} - \frac{|\nabla u|^2}{u^2} + \frac{m}{2t} \geq 0.$$

As the corollaries of Theorem 1.2 and Theorem 2.3, we have the following Harnack inequalities for positive solutions of the heat equation of the Witten Laplacian.

Corollary 1.5 *Under the same condition as in Theorem 1.2, for any $\delta > 0$, and for all $x, y \in M$, $0 < t < T$, we have*

$$u(x, t) \leq u(y, t)^{\frac{1}{1+\delta}} A^{\frac{\delta}{1+\delta}} \exp \left\{ \frac{1+\delta^{-1}}{4(1+\delta)} \frac{2K}{1-e^{-2Kt}} d^2(x, y) \right\}.$$

Corollary 1.6 *Let M be a complete Riemannian manifold with $\text{Ric}_{m,n}(L) \geq -K$, u be a positive solution to the heat equation $\partial_t u = Lu$. Then, for all $x, y \in M$, $0 < \tau < T$, we have*

$$u(x, \tau) \leq \left(\frac{T}{\tau} \right)^{m/2} u(y, T) \exp \left\{ \frac{1}{4} e^{2K\tau} \left[1 + 2K(T - \tau) \frac{d^2(x, y)}{T - \tau} + \frac{m}{2} [e^{2KT} - e^{2K\tau}] \right] \right\}.$$

In Section 3, we introduce the W -entropy and prove the W -entropy formulas for the heat equation of the Witten Laplacian on complete Riemannian manifolds satisfying the $CD(K, m)$ condition, for $K \in \mathbb{R}$ and $m \in [n, \infty]$. We also prove a rigidity theorem on complete Riemannian manifolds satisfying the $CD(K, m)$ condition for $K \in \mathbb{R}$ and $m \in [n, \infty)$. These extend the W -entropy formula proved in [15, 17, 18] for the Witten Laplacian on complete Riemannian manifolds satisfying the $CD(0, m)$ condition, $m \in [n, \infty)$. We will also extend the W -entropy formula to time dependent Witten Laplacian on compact Riemannian manifolds with K -super Perelman Ricci flow.

Theorem 1.7 *Let M be a complete Riemannian manifold with bounded geometry condition, $\phi \in C^4(M)$ with $\nabla \phi \in C_b^3(M)$. Suppose that $\text{Ric} + \nabla^2 \phi \geq K$, where $K \in \mathbb{R}$ is a constant. Let $u(\cdot, t) = P_t f$ be a positive solution to the heat equation $\partial_t u = Lu$ with $u(\cdot, 0) = f$, f is a positive and measurable function on M . Let*

$$H_K(f, t) = D_K(t) \int_M (P_t(f \log f) - P_t f \log P_t f) d\mu,$$

where $D_0(t) = \frac{1}{t}$ and $D_K(t) = \frac{1}{1-e^{-2Kt}}$ for $K \neq 0$. Then, for all $K \in \mathbb{R}$,

$$\frac{d}{dt} H_K(f, t) \leq 0, \quad \forall t > 0,$$

and for all $K \in \mathbb{R}$ and $t > 0$, we have

$$\frac{d^2}{dt^2} H_K(t) + 2K \coth(Kt) \frac{d}{dt} H_K(t) \leq -2D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu.$$

Define the W -entropy by the revised Boltzmann entropy formula

$$W_K(f, t) = H_K(f, t) + \frac{\sinh(2Kt)}{2K} \frac{d}{dt} H_K(f, t).$$

Then, for all $K \in \mathbb{R}$, and for all $t > 0$, we have

$$\begin{aligned} \frac{d}{dt} W_K(f, t) &= -\frac{\sinh(2Kt)}{K} D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu \\ &\quad - \frac{\sinh(2Kt)}{K} D_K(t) \int_M (\text{Ric}(L) - K)(\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu. \end{aligned}$$

In particular, for all $K \in \mathbb{R}$, we have

$$\frac{d}{dt} W_K(f, t) \leq 0, \quad \forall t > 0.$$

Theorem 1.8 *Let M be a complete Riemannian manifold with a fixed metric and potential (g, ϕ) . Suppose that (M, g) satisfies the bounded geometry condition and $\phi \in C^4(M)$ with $\nabla \phi \in C_b^3(M)$. Let u be the heat kernel of the Witten Laplacian $L = \Delta - \nabla \phi \cdot \nabla$. Let*

$$H_{m,K}(u, t) = - \int_M u \log u d\mu - \Phi_{m,K}(t),$$

where $\Phi_{m,K} \in C((0, \infty), \mathbb{R})$ satisfies

$$\Phi'_{m,K}(t) = \frac{m}{2t} e^{4Kt}, \quad \forall t > 0.$$

Define the W -entropy by the Boltzmann formula

$$W_{m,K}(u, t) = \frac{d}{dt} (t H_{m,K}(u, t)).$$

Then

$$\begin{aligned} \frac{d}{dt} W_{m,K}(u, t) &= -2t \int_M \left[\left| \nabla^2 \log u + \left(\frac{K}{2} + \frac{1}{2t} \right) g \right|^2 + (\text{Ric}_{m,n}(L) + Kg)(\nabla \log u, \nabla \log u) \right] u d\mu \\ &\quad - \frac{2t}{m-n} \int_M \left| \nabla \phi \cdot \nabla \log u - \frac{(m-n)(1+Kt)}{2t} \right|^2 u d\mu \\ &\quad - \frac{m}{2t} [e^{4Kt}(1+4Kt) - (1+Kt)^2]. \end{aligned}$$

In particular, if $\text{Ric}_{m,n}(L) \geq -K$, then, for all $t \geq 0$, we have

$$\frac{d}{dt} W_{m,K}(u, t) \leq -\frac{m}{2t} [e^{4Kt}(1+4Kt) - (1+Kt)^2].$$

Moreover, the equality holds at some time $t = t_0 > 0$ if and only if M is a quasi-Einstein manifold, i.e., $\text{Ric}_{m,n}(L) = -Kg$, and the potential function $f = -\log u$ satisfies the shrinking soliton equation with respect to $\text{Ric}_{m,n}(L)$, i.e.,

$$\text{Ric}_{m,n}(L) + 2\nabla^2 f = \frac{g}{t},$$

and moreover

$$\nabla \phi \cdot \nabla f = -\frac{(m-n)(1+Kt)}{2t}.$$

We would like to point out that, Theorem 1.7 and Theorem 1.8 are new even in the case ϕ is a constant, $m = n$ and $L = \Delta$ is the usual Laplace-Beltrami operator on complete Riemannian manifolds with Ricci curvature bounded from below by a negative constant. In this case, Theorem 1.8 can be formulated as follows.

Theorem 1.9 *Let (M, g) be a complete Riemannian manifold with bounded geometry condition. Then*

$$\begin{aligned} \frac{d}{dt} W_{n,K}(u, t) &= -2t \int_M u \left[\left| \nabla^2 \log u + \left(\frac{K}{2} + \frac{1}{2t} \right) g \right|^2 + (Ric + Kg)(\nabla \log u, \nabla \log u) \right] u d\mu \\ &\quad - \frac{n}{2t} [e^{4Kt}(1 + 4Kt) - (1 + Kt)^2]. \end{aligned}$$

In particular, if $Ric \geq -K$, then, for all $t \geq 0$, we have

$$\frac{d}{dt} W_{n,K}(u, t) \leq -\frac{n}{2t} [e^{4Kt}(1 + 4Kt) - (1 + Kt)^2].$$

Moreover, the equality holds at some time $t = t_0 > 0$ if and only if M is an Einstein manifold, i.e., $Ric = -Kg$, and the potential function $f = -\log u$ satisfies the shrinking soliton equation, i.e.,

$$Ric + 2\nabla^2 f = \frac{g}{t}.$$

In Subsection 3.3, we will extend Theorem 1.7 and Theorem 1.8 to time dependent Witten Laplacian on compact Riemannian manifolds with a K -super Perelman Ricci flow. For details, see Theorem 3.6 and Theorem 3.7.

In Section 4, we prove the Li-Yau Harnack inequality and the Li-Yau-Hamilton Harnack inequality for positive solutions to the heat equation $\partial_t u = Lu$ of the time dependent Witten Laplacian on compact Riemannian manifolds equipped with variants of the (K, m) -super Ricci flow.

Theorem 1.10 *Let $(M, g(t), \phi(t), t \in [0, T])$ be a compact Riemannian manifold with a family of time dependent metrics $g(t)$ and potentials $\phi(t) \in C^2(M)$, $t \in [0, T]$. Let u be a positive solution to the heat equation $\partial_t u = Lu$. Let $\partial_t g = 2h$ and $\alpha > 1$. Suppose that $(M, g(t), \phi(t), t \in [0, T])$ satisfies the backward (α, K, m) -super Ricci flow*

$$\frac{1}{2}(1 - \alpha)\partial_t g + Ric_{m,n}(L) \geq -Kg, \quad (24)$$

and assume that $A^2 = \max \left[|h|^2 + \frac{(\text{Tr} h)^2}{m-n} \right] < \infty$ and $B = \max |S| < \infty$, where

$$S(\cdot) = 2h(\nabla \phi, \cdot) - \langle 2\text{div} h - \nabla \text{Tr}_g h + \nabla \phi_t, \cdot \rangle + \frac{2\text{Tr} h}{m-n} \langle \nabla \phi, \cdot \rangle.$$

Then for any $\gamma > 0$ and for all $t \in (0, T]$, we have

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{\partial_t u}{u} \leq \frac{m\alpha^2}{4t} \left[1 + \sqrt{1 + \frac{T^2}{m} \left(4A^2 + \frac{(2K + \gamma)^2}{(\alpha - 1)^2} + \frac{2B^2}{\gamma} \right)} \right].$$

In the case $B = 0$, for all $t \in (0, T]$, we have

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{\partial_t u}{u} \leq \frac{m\alpha^2}{4t} \left[1 + \sqrt{1 + \frac{T^2}{m} \left(4A^2 + \frac{4K^2}{(\alpha - 1)^2} \right)} \right].$$

In particular, in the case $A = B = 0$ and $\text{Ric}_{m,n}(L) \geq 0$, we have the Li-Yau Harnack inequality

$$\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{m}{2t}.$$

Theorem 1.11 *Let $(M, g(t), t \in [0, T])$ be a compact Riemannian manifold with a family of time dependent metrics $g(t)$ and potentials $\phi(t) \in C^2(M)$, $t \in [0, T]$. Let u be a positive solution to the heat equation $\partial_t u = Lu$. Let $\partial_t g = 2h$. Suppose that, for all $t \in (0, T]$,*

$$e^{-4Kt} (h + \text{Ric}_{m,n}(L) + Kg) - e^{-2Kt} h \geq \alpha_K(t)g, \quad (25)$$

where $\alpha_K(t)$ is a real valued function in time t , and

$$A^2 = \max \left[|h|^2 + \frac{(\text{Tr} h)^2}{m-n} \right] < \infty, \quad B = \max |S| < \infty,$$

where

$$S(\cdot) = \left\langle \frac{2\text{Tr} h}{m-n} \nabla \phi - 2\text{div} h - \nabla \text{Tr}_g h + \nabla \partial_t \phi, \cdot \right\rangle + 2h(\nabla \phi, \cdot).$$

Then, for any $\gamma > 0$ and $t \in [0, T]$, we have

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt} \frac{\partial_t u}{u} \leq \frac{me^{4Kt}}{2t} \left[1 + \sqrt{\frac{A^2 T^2}{m} + \max_{t \in [0, T]} \frac{t^2 (2\alpha_K(t) - \gamma)^2}{4e^{-4Kt}(1 - e^{-2Kt})^2} + \max_{t \in [0, T]} \frac{t^2 e^{-4Kt} B^2}{2m\gamma}} \right].$$

In the case $B = 0$, we have

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt} \frac{\partial_t u}{u} \leq \frac{me^{4Kt}}{2t} \left[1 + \sqrt{\frac{A^2 T^2}{m} + \max_{t \in [0, T]} \frac{t^2 \alpha_K^2(t)}{e^{-4Kt}(1 - e^{-2Kt})^2}} \right].$$

and if $\alpha_K(t) = 0$, i.e., if

$$e^{-4Kt} (h + \text{Ric}_{m,n}(L) + K) - e^{-2Kt} h \geq 0, \quad (26)$$

we have

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt} \frac{\partial_t u}{u} \leq \frac{me^{4Kt}}{2t} \left[1 + \frac{TA}{\sqrt{m}} \right].$$

In particular, when $A = B = 0$, and $\text{Ric}_{m,n}(L) \geq -K$, we recapture Hamilton's Harnack inequality [10]

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt} \frac{\partial_t u}{u} \leq \frac{me^{4Kt}}{2t}.$$

Theorem 1.10 and Theorem 1.11 can be also extended to complete Riemannian manifolds equipped with variants of the (K, m) -super Ricci flow. For details, see Section 5.

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.1, Theorem 1.2, Corollary 1.5, Theorem 2.3 and Corollary 1.6. In Section 3, we prove Theorem 1.7, Theorem 1.8 and extend Theorem 1.7 and Theorem 1.8 to time dependent Witten Laplacian on compact Riemannian manifolds with a K -super Perelman Ricci flow, see Theorem 3.6 and Theorem 3.7. In Section 4, we prove Theorem 1.10 and Theorem 1.11. In Section 5, we extend Theorem 1.10 and Theorem 1.11 to complete Riemannian manifolds equipped with variants of the (K, m) -super Ricci flows.

2 Log-Sobolev inequalities and Harnack inequalities for Witten Laplacian

2.1 Log-Sobolev inequalities on K -super Perelman Ricci flow

In this subsection, we modify the semigroup argument due to Bakry and Ledoux [2] to prove the equivalence between the K -super Perelman Ricci flow equation and two logarithmic Sobolev inequalities for the time dependent Witten Laplacian.

Proof. Let $P_{s,t}$ be the heat semigroup of the time dependent weighted Laplacian on $L^2(M, \mu)$, i.e., for any $s \in [0, T]$, $u(t, \cdot) := P_{s,t}f(\cdot)$ is the unique solution of the heat equation $\partial_t u = Lu$ in $L^2(M, \mu)$ on $[s, T]$ with $u(s, \cdot) = f$. Let

$$h(s, t) = e^{2Kt} P_{s+T-t, T} \left(\frac{|\nabla P_{s, s+T-t} f|^2}{P_{s, s+T-t} f} \right), \quad t \in [s, T].$$

Note that, at time $T - t + s$, the generalized Bochner formula implies

$$(\partial_t + L) \frac{|\nabla u|^2}{u} = \frac{2}{u} |\nabla^2 u - u^{-1} \nabla u \otimes \nabla u|^2 + 2u^{-1} \left(\frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) \right) (\nabla u, \nabla u). \quad (27)$$

Hence

$$\begin{aligned} \partial_t h(s, t) &= 2Kh(s, t) + e^{2Kt} P_{s+T-t, T} \left[\left(\frac{\partial}{\partial t} + L \right) \left(\frac{|\nabla P_{s, s+T-t} f|^2}{P_{s, s+T-t} f} \right) \right] \\ &= 2Kh(s, t) + e^{2Kt} P_{s+T-t, T} \left[\frac{2}{u} |\nabla^2 u - u^{-1} \nabla u \otimes \nabla u|^2 + 2u^{-1} \left(\frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) \right) (\nabla u, \nabla u) \right] \\ &\geq 2e^{2Kt} P_{s+T-t, T} \left[u^{-1} \left(\frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) + K \right) (\nabla u, \nabla u) \right]. \end{aligned}$$

If $(g(t), \phi(t))$ is a (K, ∞) -super Ricci flow, i.e., if $\frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) + K \geq 0$, then

$$\partial_t h(s, t) \geq 0.$$

Thus, $t \rightarrow h(s, t)$ is increasing on $[s, T]$. This yields, for all $t \in (s, T)$,

$$e^{2Ks} \frac{|\nabla P_{s, T} f|^2}{P_{s, T} f} \leq e^{2Kt} P_{s+T-t, T} \left(\frac{|\nabla P_{s, s+T-t} f|^2}{P_{s, s+T-t} f} \right) \leq e^{2KT} P_{s, T} \left(\frac{|\nabla f|^2}{f} \right).$$

Notice that

$$\begin{aligned} \frac{d}{dt} P_{s+T-t, T} (P_{s, s+T-t} f \log P_{s, s+T-t} f) &= P_{s+T-t, T} ((L_{s+T-t} + \partial_t)(P_{s, s+T-t} f \log P_{s, s+T-t} f)) \\ &= P_{s+T-t, T} \left(\frac{|\nabla P_{s, s+T-t} f|^2}{P_{s, s+T-t} f} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} P_{s, T} (f \log f) - P_{s, T} f \log P_{s, T} f &= \int_s^T \frac{d}{dt} P_{s+T-t, T} (P_{s, s+T-t} f \log P_{s, s+T-t} f) dt \\ &= \int_s^T P_{s+T-t, T} \left(\frac{|\nabla P_{s, s+T-t} f|^2}{P_{s, s+T-t} f} \right) dt \\ &\leq \frac{1}{2K} (e^{2K(T-s)} - 1) P_{s, T} \left(\frac{|\nabla f|^2}{f} \right). \end{aligned}$$

Thus the logarithmic Sobolev inequality holds on complete Riemannian manifolds equipped with a K -super Perelman Ricci flow

$$P_{s,T}(f \log f) - P_{s,T} f \log P_{s,T} f \leq \frac{e^{2K(T-s)} - 1}{2K} P_{s,T} \left(\frac{|\nabla f|^2}{f} \right). \quad (28)$$

Similarly to the above proof of the logarithmic Sobolev inequality (28), we have

$$\begin{aligned} P_{s,T}(f \log f) - P_{s,T} f \log P_{s,T} f &= \int_s^T \frac{d}{dt} P_{s+T-t,T} (P_{s,s+T-t} f \log P_{s,s+T-t} f) dt \\ &= \int_s^T P_{s+T-t,T} \left(\frac{|\nabla P_{s,s+T-t} f|^2}{P_{s,s+T-t} f} \right) dt \\ &\geq \int_s^T e^{2K(s-t)} \frac{|\nabla P_{s,T} f|^2}{P_{s,T} f} dt \\ &= \frac{1 - e^{2K(s-T)}}{2K} \frac{|\nabla P_{s,T} f|^2}{P_{s,T} f}. \end{aligned}$$

Thus, for all $T > 0$, $f \in C_b(M)$ with $f > 0$, the reversal logarithmic Sobolev inequality holds

$$\frac{|\nabla P_{s,T} f|^2}{P_{s,T} f} \leq \frac{2K}{1 - e^{2K(s-T)}} (P_{s,T}(f \log f) - P_{s,T} f \log P_{s,T} f). \quad (29)$$

Changing T by t , we see that (28) and (29) hold for $P_{s,t}$ for all $0 \leq s < t \leq T$.

On the other hand, if for all $0 \leq s < t \leq T$, the log-Sobolev inequality holds for $P_{s,t}$, then applying (28) to $1 + \varepsilon f$ and letting $\varepsilon \rightarrow 0$, we can obtain the Poincaré inequality

$$P_{s,t} f^2 - (P_{s,t} f)^2 \leq \frac{1}{K} (e^{2K(t-s)} - 1) P_{s,t} (|\nabla f|^2).$$

Set

$$w(s, t) = P_{s,t} f^2 - (P_{s,t} f)^2 - \frac{1}{K} (e^{2K(t-s)} - 1) P_{s,t} (|\nabla f|^2).$$

Then $w(s, t) \leq 0$ for all $0 \leq s < t < T$. Notice that when $s = t$, we have $w(s, s) = 0$. Hence, for all $s < t$, $\partial_s w(s, t) \leq 0$. Now

$$\begin{aligned} \partial_s w(s, t) &= -P_{s,t} L_s f^2 + 2P_{s,t} f P_{s,t} L_s f + 2e^{2K(t-s)} P_{s,t} (|\nabla f|^2) \\ &\quad - \frac{e^{2K(t-s)} - 1}{K} [P_{s,t} (-L(|\nabla f|^2) + \partial_s |\nabla f|_{g(s)}^2)]. \end{aligned}$$

Thus, at $s = t$, we have $\partial_s w(s, t)|_{s=t} = 0$. On the other hand, as for all $s < t$, $\partial_s w(s, t) \leq 0$, we can derive that $\partial_s^2 w(s, t) \leq 0$ for all $0 \leq s < t < T$. By calculation, at $s = t$, we have

$$\begin{aligned} \partial_s^2 w(s, t) &= -\partial_s (P_{s,t} L_s f^2) + 2\partial_s (P_{s,t} f P_{s,t} L_s f) - 4K e^{2K(t-s)} P_{s,t} (|\nabla f|^2) \\ &\quad + 2e^{2K(t-s)} \partial_s P_{s,t} (|\nabla f|^2) + 2e^{2K(t-s)} P_{s,t} (-L_s (|\nabla f|^2) + \partial_s |\nabla f|_{g(s)}^2) \\ &= L_s^2 f^2 - \partial_s (L_s f^2) - 2(L_s f)^2 - 2f L_s^2 f + 2f \partial_s L_s f \\ &\quad - 4K (|\nabla f|^2) - 4L_s (|\nabla f|^2) + 4\partial_s |\nabla f|_{g(s)}^2. \end{aligned}$$

Note that

$$\begin{aligned} L_s^2 f^2 &= L_s (2f L_s f + 2|\nabla f|_{g(s)}^2) \\ &= 2L_s f L_s f + 2f L_s^2 f + 4\nabla f \cdot \nabla L_s f + 2L_s |\nabla f|_{g(s)}^2, \\ \partial_s L_s f^2 &= \partial_s (2f L_s f + 2|\nabla f|_{g(s)}^2) = 2f \partial_s L_s f + 2\partial_s |\nabla f|_{g(s)}^2, \end{aligned}$$

and

$$\partial_s |\nabla f|_{g(s)}^2 = -\partial_s g(s)(\nabla f, \nabla f),$$

from which and the Bochner formula, we see that at $s = t$,

$$\begin{aligned} \partial_s^2 w(s, t) &= -2L_s |\nabla f|_{g(s)}^2 + 4\nabla f \cdot \nabla L_s f - 4K |\nabla f|_{g(s)}^2 + 2\partial_s |\nabla f|_{g(s)}^2 \\ &= -4 \left[\|\text{Hess} f\|_{g(s)}^2 + \left(\frac{1}{2} \partial_s g(s) + \text{Ric}(L_s) + K \right) (\nabla f, \nabla f) \right] \Big|_{s=t}. \end{aligned}$$

Taking f to be normal coordinate functions near x on $(M, g(t))$, we derive that, at any time $t \in [0, T]$,

$$\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L_t) + K \geq 0.$$

This completes the proof of Theorem 1.1.

Remark 2.1 Indeed, we can further prove the Poincaré inequality, the reversal Poincaré inequality as well as Bakry-Ledoux's Gromov-Lévy isoperimetric inequality on the super Ricci flow (19). To save the length of the paper, we will do these in a forthcoming paper. In [24, 25, 26], Sturm introduced the super Ricci flow on metric measure space, and proved the equivalence between the super Ricci flow and the Poincaré inequality.

2.2 Hamilton's Harnack inequality for Witten Laplacian with $CD(K, \infty)$ condition

In this subsection, we prove Hamilton's Harnack inequality for the time dependent Witten Laplacian on complete Riemannian manifolds with a K -super Perelman Ricci flow.

Proof of Theorem 1.2. We modify the method used in [17]. Let $t \in [0, T]$ and $s \in [0, T-t]$. Using the reversal logarithmic Sobolev inequality and the fact $0 < f \leq A$, we have

$$\begin{aligned} \frac{|\nabla P_{s,s+t} f|^2}{P_{s,s+t} f} &\leq \frac{2K}{1 - e^{-2Kt}} (P_{s,s+t}(f \log f) - P_{s,s+t} f \log P_{s,s+t} f) \\ &\leq \frac{2K}{1 - e^{-2Kt}} (P_{s,s+t}(f \log A) - P_{s,s+t} f \log P_{s,s+t} f). \end{aligned}$$

Thus

$$|\nabla \log P_{s,s+t} f|^2 \leq \frac{2K}{1 - e^{-2Kt}} \log(A/P_{s,s+t} f).$$

Using $\frac{1}{1-e^{-x}} \leq 1 + \frac{1}{x}$ for $x \geq 0$, we have

$$|\nabla \log P_{s,s+t} f|^2 \leq \left(2K + \frac{1}{t} \right) \log(A/P_{s,s+t} f).$$

In particular, for $s = 0$, we have

$$\frac{|\nabla u|^2}{u^2} \leq \left(2K + \frac{1}{t} \right) \log(A/u).$$

The proof of Theorem 1.2 is completed. \square

Proof of Corollary 1.5. Let $l(x, t) = \log A/u(x, t)$. Then the differential Harnack inequality (21) in Theorem 1.2 implies

$$|\nabla \sqrt{l(x, t)}| = \frac{1}{2} \frac{|\nabla l(x, t)|}{\sqrt{l(x, t)}} \leq \frac{1}{2} \sqrt{\frac{2K}{1 - e^{-2Kt}}}.$$

Fix $x, y \in M$ and integrate along a geodesic linking x and y , the above inequality yields

$$\sqrt{\log A/u(x, t)} \leq \sqrt{\log A/u(y, t)} + \frac{1}{2} \sqrt{\frac{2K}{1 - e^{-2Kt}}} d(x, y).$$

Combining this with the elementary inequality

$$(a + b)^2 \leq (1 + \delta)a^2 + (1 + \delta^{-1})b^2,$$

we can derive the desired Harnack inequality for u . \square

2.3 The LYH Harnack inequality for Witten Laplacian with $CD(K, m)$ condition

In this subsection we prove Theorem , i.e, the Li-Yau-Hamilton Harnack inequality for the positive solution to the heat equation associated with the Witten Laplacian on complete Riemannian manifold with fixed metric and potential. Indeed, by the generalized Bochner-Weitzenböck formula, we have

$$(L - \partial_t) \frac{|\nabla u|^2}{u} = \frac{2}{u} \left| \nabla^2 u - \frac{\nabla u \otimes \nabla u}{u} \right|^2 + \frac{2}{u} Ric(L)(\nabla u, \nabla u). \quad (30)$$

Taking trace in the first quantity on the right hand side, we can derive

$$(L - \partial_t) \frac{|\nabla u|^2}{u} \geq \frac{2}{nu} \left| \Delta u - \frac{|\nabla u|^2}{u} \right|^2 + \frac{2}{u} Ric(L)(\nabla u, \nabla u). \quad (31)$$

Applying the inequality

$$(a + b)^2 \geq \frac{a^2}{1 + \alpha} - \frac{b^2}{\alpha}$$

to $a = \partial_t u - \frac{|\nabla u|^2}{u}$, $b = \nabla \phi \cdot \nabla u$, and $\alpha = \frac{m-n}{n}$, we have

$$(L - \partial_t) \frac{|\nabla u|^2}{u} \geq \frac{2}{mu} \left| \partial_t u - \frac{|\nabla u|^2}{u} \right|^2 + \frac{2}{u} Ric_{m,n}(L)(\nabla u, \nabla u). \quad (32)$$

Hence, under the condition $Ric_{m,n}(L) \geq -K$, it holds

$$(L - \partial_t) \frac{|\nabla u|^2}{u} \geq \frac{2}{mu} \left| \partial_t u - \frac{|\nabla u|^2}{u} \right|^2 - \frac{2K|\nabla u|^2}{u}. \quad (33)$$

Let

$$h = \frac{\partial u}{\partial t} - e^{-2Kt} \frac{|\nabla u|^2}{u} + e^{2Kt} \frac{m}{2t} u.$$

Then $\lim_{t \rightarrow 0^+} h(t) = +\infty$, and

$$(\partial_t - L)h \geq \frac{2}{mu} e^{-2Kt} \left| \partial_t u - \frac{|\nabla u|^2}{u} \right|^2 - e^{2Kt} \frac{m}{2t^2} u.$$

We now prove that $h \geq 0$ on $M \times \mathbb{R}^+$. In compact case, suppose that h attains its minimum at some (x_0, t_0) and $h(x_0, t_0) < 0$. Then, at (x_0, t_0) , it holds

$$\frac{\partial h}{\partial t} \leq 0, \quad \Delta h \geq 0, \quad \nabla h = 0.$$

Thus at (x_0, t_0) , $(\partial_t - L)h \leq 0$. On the other hand, at this point, we have

$$0 \leq e^{2Kt} \frac{m}{2t} u < e^{-2Kt} \frac{|\nabla u|^2}{u} - \frac{\partial u}{\partial t} \leq \frac{|\nabla u|^2}{u} - \frac{\partial u}{\partial t},$$

and hence

$$(\partial_t - L)h > 0.$$

This finishes the proof of Theorem 2.3 in compact case.

In complete non-compact case, let $f = \log u$, and let

$$F = te^{-2Kt}(e^{-2Kt}|\nabla f|^2 - f_t) = te^{-4Kt}|\nabla f|^2 - te^{-2Kt}f_t.$$

Obviously, $F(0, x) \equiv 0$. We shall prove that

$$F \leq \frac{m}{2}.$$

By direct calculation

$$\begin{aligned} LF &= te^{-4Kt}L|\nabla f|^2 - te^{-2Kt}Lf_t \\ \partial_t F &= \partial_t(te^{-4Kt}|\nabla f|^2 - te^{-2Kt}f_t) \\ &= (1 - 4Kt)e^{-4Kt}|\nabla f|^2 + (2Kt - 1)e^{-2Kt}f_t + te^{-4Kt}\partial_t|\nabla f|^2 - te^{-2Kt}f_{tt}, \end{aligned}$$

we have

$$\begin{aligned} (L - \partial_t)F &= te^{-4Kt}(L - \partial_t)|\nabla f|^2 - te^{-2Kt}(L - \partial_t)f_t \\ &\quad + (4Kt - 1)e^{-4Kt}|\nabla f|^2 - (2Kt - 1)e^{-2Kt}f_t. \end{aligned}$$

By the generalized Bochner formula, it holds

$$(L - \partial_t)|\nabla f|^2 = 2|\nabla^2 f|^2 + 2Ric(L)(\nabla f, \nabla f) - 4\nabla^2 f(\nabla f, \nabla f).$$

Note that

$$\begin{aligned} Lf_t &= L\left(\frac{Lu}{u}\right) = \frac{L^2u}{u} - 2\langle \nabla Lu, \frac{\nabla u}{u^2} \rangle + Lu\left(-\frac{Lu}{u^2} + 2\frac{|\nabla u|^2}{u^3}\right), \\ \partial_t f_t &= \partial_t\left(\frac{Lu}{u}\right) = \frac{L^2u}{u} - \frac{|Lu|^2}{u^2}, \end{aligned}$$

which yields

$$\begin{aligned} (L - \partial_t)f_t &= 2\frac{Lu|\nabla u|^2}{u^3} - 2\langle \nabla Lu, \frac{\nabla u}{u^2} \rangle \\ &= -4\nabla^2 f(\nabla f, \nabla f) - 2\langle \nabla Lf, \nabla f \rangle. \end{aligned}$$

Hence

$$\begin{aligned} (L - \partial_t)F &= 2te^{-4Kt}[|\nabla^2 f|^2 + 2(e^{2Kt} - 1)\nabla^2 f(\nabla f, \nabla f)] \\ &\quad + 2te^{-4Kt}Ric(L)(\nabla f, \nabla f) + 2te^{-2Kt}\langle \nabla Lf, \nabla f \rangle \\ &\quad + (4Kt - 1)e^{-4Kt}|\nabla f|^2 - (2Kt - 1)e^{-2Kt}(Lf + |\nabla f|^2). \end{aligned}$$

Now

$$\begin{aligned} F &= te^{-4Kt}|\nabla f|^2 - te^{-2Kt}f_t = te^{-4Kt}(1 - e^{2Kt})|\nabla f|^2 - te^{-2Kt}Lf, \\ \langle \nabla F, \nabla f \rangle &= 2te^{-4Kt}(1 - e^{2Kt})\nabla^2 f(\nabla f, \nabla f) - te^{-2Kt}\langle \nabla Lf, \nabla f \rangle. \end{aligned}$$

Therefore

$$\begin{aligned}(L - \partial_t)F &= 2te^{-4Kt}|\nabla^2 f|^2 - 2\langle \nabla F, \nabla f \rangle \\ &\quad + 2te^{-4Kt}(\text{Ric}(L)(\nabla f, \nabla f) + K|\nabla f|^2) + \frac{(2Kt-1)}{t}F.\end{aligned}$$

By [13], we have

$$|\nabla^2 f|^2 \geq \frac{1}{n}|\Delta f|^2 \geq \frac{1}{m}|\nabla f|^2 - \frac{1}{m-n}\nabla\phi \otimes \nabla\phi(\nabla f, \nabla f).$$

Thus

$$\begin{aligned}(L - \partial_t)F &\geq 2te^{-4Kt}\frac{|\nabla f|^2}{m} - 2\langle \nabla F, \nabla f \rangle \\ &\quad + 2te^{-4Kt}(\text{Ric}_{m,n}(L)(\nabla f, \nabla f) + K|\nabla f|^2) + \frac{(2Kt-1)}{t}F \\ &\geq \frac{2te^{-4Kt}}{m} \left[\frac{(te^{-2Kt}(e^{-2Kt}-1)|\nabla f|^2 - F)^2}{t^2e^{-4Kt}} \right] - 2\langle \nabla F, \nabla f \rangle + \frac{(2Kt-1)}{t}F \\ &\geq \frac{2[te^{-2Kt}(e^{-2Kt}-1)|\nabla f|^2 - F]^2}{mt} - 2\langle \nabla F, \nabla f \rangle + \frac{(2Kt-1)}{t}F.\end{aligned}$$

Similarly to [13], let η be a C^2 -function on $[0, \infty)$ such that $\eta = 1$ on $[0, 1]$ and $\eta = 0$ on $[2, \infty)$, with $-C_1\eta^{1/2}(r) \leq \eta'(r) \leq 0$, and $\eta''(r) \geq C_2$, where $C, C_2 > 0$ are two constants. Let $\rho(x) = d(o, x)$ and define $\psi(x) = \eta(\rho(x)/R)$. Since ρ is Lipschitz on the complement of the cut locus of o , ψ is a Lipschitz function with support in $B(o, 2R) \times [0, \infty)$. As explained in Li and Yau [43], an argument of Calabi allows us to apply the maximum principle to ψF . Let $(x_0, t_0) \in M \times [0, T]$ be a point where ψF achieves the maximum. Then, at (x_0, t_0) ,

$$\partial_t(\psi F) \geq 0, \quad \Delta(\psi F) \leq 0, \quad \nabla(\psi F) = 0.$$

This yields

$$(L - \partial_t)(\psi F) = \Delta(\psi F) - \nabla\phi \cdot \nabla(\psi F) - \partial_t(\psi F) \leq 0.$$

Similarly to [13], we have

$$\begin{aligned}(L - \partial_t)(\psi F) &= \psi(L - \partial_t)F + (L\psi)F + 2\nabla\psi \cdot \nabla F \\ &\geq \psi(L - \partial_t)F - A(R)F + 2\nabla\psi \cdot \nabla F \\ &\geq \psi(L - \partial_t)F - A(R)F + 2\langle \nabla\psi, \nabla(\psi F) \rangle \psi^{-1} - 2F|\nabla\psi|^2 \psi^{-1}.\end{aligned}$$

where we use

$$L\psi \geq -A(R) := -\frac{C_1}{R}(m-1)\sqrt{K} \coth(\sqrt{K}R) - \frac{C_2}{R^2},$$

and for some constant $C_3 > 0$

$$\frac{|\nabla\psi|^2}{\psi} \leq \frac{C_3}{R^2}.$$

Let $C(n, K, R) = \frac{C_1}{R}(m-1)\sqrt{K} \coth(\sqrt{K}R) + \frac{C_2+C_3}{R^2}$. At the point (x_0, t_0) , we have

$$\begin{aligned}
0 &\geq \psi(L - \partial_t)F - (A(R) + 2|\nabla\psi|^2\psi^{-1})F \\
&\geq \psi \left[\frac{2[te^{-2Kt}(e^{-2Kt} - 1)|\nabla f|^2 - F]^2}{mt} - 2\langle \nabla F, \nabla f \rangle + \frac{(2Kt-1)}{t}F \right] - C(n, K, R)F \\
&\geq \psi \frac{2}{mt}F^2 + \psi \frac{4e^{-2Kt}(1 - e^{-2Kt})|\nabla f|^2}{m}F + 2F\langle \nabla\psi, \nabla f \rangle + \left[(2K - \frac{1}{t})\psi - C(n, K, R) \right] F \\
&\geq \psi \frac{2}{mt}F^2 + \psi \frac{4e^{-2Kt}(1 - e^{-2Kt})|\nabla f|^2}{m}F - 2F|\nabla\psi||\nabla f| + \left[(2K - \frac{1}{t})\psi - C(n, K, R) \right] F \\
&\geq \psi \frac{2}{mt}F^2 + \psi \frac{4e^{-2Kt}(1 - e^{-2Kt})|\nabla f|^2}{m}F - 2\frac{C_2}{R}F\psi^{1/2}|\nabla f| + \left[(2K - \frac{1}{t})\psi - C(n, K, R) \right] F.
\end{aligned}$$

Multiplying by t on both sides, and using the Cauchy-Schwartz inequality, we get

$$\begin{aligned}
0 &= \psi \frac{2}{m}F^2 + tF \left[\psi \frac{4e^{-2Kt}(1 - e^{-2Kt})|\nabla f|^2}{m} - 2\frac{C_2}{R}\psi^{1/2}|\nabla f| \right] + [(2Kt-1)\psi - C(n, K, R)t]F \\
&\geq \psi \frac{2}{m}F^2 + \left[(2Kt-1)\psi - C(n, K, R)t - \frac{C_2m}{4e^{-2Kt}(1 - e^{-2Kt})R^2}t \right] F.
\end{aligned}$$

Notice that the above calculation is done at the point (x_0, t_0) . Since ψF reaches its maximum at this point, we can assume that $\psi F(x_0, t_0) > 0$. Thus

$$0 \geq \frac{2}{m}(\psi F)^2 - \left[1 + C(n, K, R)t + \frac{C_2m}{4e^{-2Kt}(1 - e^{-2Kt})R^2}t \right] (\psi F),$$

which yields that, for any $(x, t) \in B_R \times [0, T]$,

$$\begin{aligned}
F(x, t) &\leq (\psi F)(x_0, t_0) \leq \frac{m}{2} \left[1 + C(n, K, R)t_0 + \frac{C_2m}{4e^{-2Kt_0}(1 - e^{-2Kt_0})R^2}t_0 \right] \\
&\leq \frac{m}{2} \left[1 + C(n, K, R)T + \max_{t \in [0, T]} \frac{C_2mt}{4e^{-2Kt}(1 - e^{-2Kt})R^2} \right].
\end{aligned}$$

Let $R \rightarrow \infty$, we obtain

$$F \leq \frac{m}{2}.$$

□

Proof of Corollary 1.6. The proof is as the same as the one of Corollary 2.2 in [10]. For the completeness we reproduce it as follows. Let $l(x, t) = \log u(x, t)$. Then the Li-Yau-Hamilton Harnack inequality is equivalent to

$$\frac{\partial l}{\partial t} - e^{-2Kt}|\nabla l|^2 + e^{2Kt}\frac{m}{2t} \geq 0. \quad (34)$$

Let $\gamma : [0, T] \rightarrow M$ be a geodesic with reparametrization by arc length $s : [\tau, T] \rightarrow [0, T]$ so that $\gamma(s(\tau)) = x$ and $\gamma(s(T)) = y$. Let $S(t) = \frac{d\gamma(s(t))}{dt} = \dot{\gamma}(s(t))\dot{s}(t)$. Then $|\dot{\gamma}(s(t))| = 1$. Integrating along $\gamma(s(t))$ from $t = \tau$ to $t = T$, we have

$$l(y, T) - l(x, \tau) = \int_{\tau}^T \left[\frac{\partial l}{\partial t} + \nabla l \cdot S \right] dt.$$

By the Cauchy-Schwartz inequality

$$e^{-2Kt}|\nabla l|^2 + \frac{1}{4}e^{2Kt}|S|^2 \geq \nabla l \cdot S$$

From this and (34) we obtain

$$l(y, T) - l(x, \tau) \geq -\frac{1}{4} \int_{\tau}^T e^{2Kt} |S|^2 dt - \int_{\tau}^T \frac{m}{2t} e^{2Kt} dt.$$

Note that $d(x, y) = \int_{\tau}^T |S| dt = \int_{\tau}^T ds(t)$. Choosing $s(t) = a[e^{-2K\tau} - e^{-2Kt}]$, with

$$a = \frac{d(x, y)}{e^{-2K\tau} - e^{-2KT}},$$

we have

$$\begin{aligned} l(y, T) - l(x, \tau) &\geq -\frac{1}{4} \int_{\tau}^T e^{2Kt} \dot{s}^2(t) dt - \int_{\tau}^T \frac{m}{2t} e^{2Kt} dt \\ &= -\frac{K d^2(x, y)}{2(e^{-2K\tau} - e^{-2KT})} - \int_{\tau}^T \frac{m}{2t} e^{2Kt} dt. \end{aligned}$$

Note that $\int_{\tau}^T \frac{e^{2Kt}}{t} dt \leq \log\left(\frac{T}{\tau}\right) + e^{2KT} - e^{2K\tau}$. Thus

$$\log u(y, T) - \log u(x, \tau) \geq -\frac{K d^2(x, y)}{2(e^{-2K\tau} - e^{-2KT})} - \frac{m}{2} \left[\log\left(\frac{T}{\tau}\right) + e^{2KT} - e^{2K\tau} \right].$$

Using $\frac{1}{1-e^{-x}} \leq \frac{1+x}{x}$, we can derive the desired estimate. \square

2.4 Hamilton's second order estimates for Witten Laplacian with $CD(K, m)$ condition

In [10], Hamilton also proved that, on compact Riemannian manifolds with $Ric \geq -K$, there exists a constant C depending only on n and K , such that for any positive solution of the heat equation $\partial_t u = \Delta u$ with $0 < u \leq A$ and $t \in [0, 1]$, it holds

$$t\Delta u \leq Cu [1 + \log(A/u)]. \quad (35)$$

Indeed, Hamilton [10] proved the following estimate

$$\frac{\Delta u}{u} + \frac{|\nabla u|^2}{u^2} \leq \frac{K}{1 - e^{-Kt}} [n + \log(A/u)], \quad \forall t \geq 0. \quad (36)$$

In this subsection, we extend Hamilton's second order estimate (36) to positive solutions of the heat equation $\partial_t u = Lu$ for the Witten Laplacian on complete Riemannian manifolds with $CD(K, m)$ -condition. More precisely, we prove the following

Theorem 2.2 *Let $m \geq n$ and $K \geq 0$ be two constants. Let M be a complete Riemannian manifold with a C^2 -potential such that the $CD(-K, m)$ condition holds*

$$Ric_{m,n}(L) \geq -K.$$

Let u be a positive solution of the heat equation

$$\partial_t u = Lu$$

with $A = \sup\{u(x, t), (x, t) \in M \times [0, T]\} < \infty$. Then

$$\frac{Lu}{u} + \frac{|\nabla u|^2}{u^2} \leq \frac{K}{1 - e^{-Kt}} [m + 4 \log(A/u)], \quad \forall t \in [0, T]. \quad (37)$$

In particular, for $t \in [0, T]$, we have

$$\frac{Lu}{u} \leq \left(K + \frac{1}{t}\right) [m + \log(A/u)]. \quad (38)$$

Proof. Let $\psi(t) = \frac{1-e^{-Kt}}{K}$. Then $\psi' + K\psi = 1$. Let $h = \psi \left[Lu + \frac{|\nabla u|^2}{u} \right] - u[m + 4 \log(A/u)]$. By (32), under the assumption $Ric_{m,n}(L) \geq -K$ we have

$$(\partial_t - L) \frac{|\nabla u|^2}{u} \leq -\frac{2}{mu} \left| Lu - \frac{|\nabla u|^2}{u} \right|^2 + 2K \frac{|\nabla u|^2}{u},$$

which yields

$$(\partial_t - L)h \leq -\frac{2\psi}{mu} \left| Lu - \frac{|\nabla u|^2}{u} \right|^2 + \psi' \left[Lu - \frac{|\nabla u|^2}{u} \right] - 2\frac{|\nabla u|^2}{u}.$$

By analogue of Hamilton[10], we can verify that

$$\frac{\partial h}{\partial t} \leq Lh \quad \text{whenever } h \geq 0.$$

Indeed, we can verify this by examining three cases:

- (i) If $Lu \leq \frac{|\nabla u|^2}{u}$, then $(\partial_t - L)h \leq 0$ since $\psi' \geq 0$.
- (ii) If $\frac{|\nabla u|^2}{u} \leq Lu \leq 3\frac{|\nabla u|^2}{u}$, then $(\partial_t - L)h \leq 0$ since $\psi' \leq 1$.
- (iii) If $3\frac{|\nabla u|^2}{u} \leq Lu$, then whenever $h \geq 0$, we have

$$2 \left[Lu - \frac{|\nabla u|^2}{u} \right] \geq Lu + \frac{|\nabla u|^2}{u} = \frac{h}{\psi} + \frac{mu + 4u \log(A/u)}{\psi} \geq \frac{mu}{\psi},$$

which yields, since $\psi' \leq 1$, we have

$$(\partial_t - L)h \leq (\psi' - 1) \left[Lu - \frac{|\nabla u|^2}{u} \right] - 2\frac{|\nabla u|^2}{u} \leq 0.$$

Note that $h \leq 0$ at $t = 0$. By the maximum principle, we conclude that $h \leq 0$ for all $t \geq 0$. Thus

$$\frac{Lu}{u} + \frac{|\nabla u|^2}{u^2} \leq \frac{K}{1 - e^{-Kt}} [m + 4 \log(A/u)].$$

This completes the proof of Theorem 2.2. \square

3 The W -entropy formula for Witten Laplacian

Recall that, Perelman [23] introduced the notion of the W -entropy and proved its monotonicity along the conjugate heat equation associated to the Ricci flow. In [21, 22], Ni proved the monotonicity of the W -entropy for the heat equation of the usual Laplace-Beltrami operator on complete Riemannian manifolds with non-negative Ricci curvature. In [15, 17], the second author of this paper proved the W -entropy formula and its monotonicity and rigidity theorems for the heat equation of the Witten Laplacian on complete Riemannian manifolds satisfying the $CD(0, m)$ condition and gave a probabilistic interpretation of the W -entropy for the Ricci flow. In [18], we gave a new proof of the W -entropy formula obtained in [15] for the Witten Laplacian by using Ni's W -entropy formula (10) to the Laplace-Beltrami operator on $M \times S^{m-n}$ equipped with a suitable warped product Riemannian metric, and further proved the monotonicity of the W -entropy for the heat equation of the time dependent Witten Laplacian on compact Riemannian manifolds equipped with the super Ricci flow with respect to the m -dimensional Bakry-Emery Ricci curvature.

During the past years, many people have asked the following very natural problem to us.

Problem 3.1 *How to define the W -entropy for the heat equation associated with the Witten Laplacian on complete Riemannian manifolds satisfying the $CD(K, m)$ condition for $K \in \mathbb{R}$ and $m \in [n, \infty]$? Can we establish the monotonicity and rigidity theorems for the W -entropy associated with the Witten Laplacian on complete Riemannian manifolds satisfying general curvature-dimension condition? What happens on manifolds with time dependent metrics and potentials?*

In this section, we give the answer to this fundamental problem.

3.1 W -entropy for Witten Laplacian with $CD(K, \infty)$ condition

In this subsection, based on the reversal logarithmic Sobolev inequality on complete Riemannian manifolds with fixed metrics and potentials, which is due to Bakry and Ledoux [2], we introduce the W -entropy and prove the W -entropy formula for the Witten Laplacian on complete Riemannian manifolds with fixed metric and potential satisfying the $CD(K, \infty)$ condition.

Let $C_0(t) = \frac{1}{t}$, and for $K \neq 0$, $C_K(t) = \frac{2K}{e^{2Kt}-1}$. Let $D_0(t) = \frac{1}{t}$, $D_K(t) = \frac{2|K|}{|1-e^{-2Kt}|}$. Then $D'_K(t) = -C_K(t)D_K(t)$ for all $K \in \mathbb{R}$ and $t > 0$. We first introduce the revised Boltzmann-Shannon entropy

$$H_K(f, t) = D_K(t) \int_M (P_t(f \log f) - P_t f \log P_t f) d\mu,$$

where f is a positive and measurable function on M . Based on the gradient estimates of the positive solution to the heat equation on complete manifolds with bounded geometry condition (see [15, 17]), by direct calculation and using the integration by parts formula, we can prove

$$\begin{aligned} \frac{d}{dt} H_K(f, t) &= C_K(t) D_K(t) \int_M (P_t f \log P_t f - P_t(f \log f)) d\mu + D_K(t) \int_M \frac{|\nabla P_t f|^2}{P_t f} d\mu \\ &= D_K(t) \int_M \left[\frac{|\nabla P_t f|^2}{P_t f} + C_K(t) (P_t f \log P_t f - P_t(f \log f)) \right] d\mu. \end{aligned} \quad (39)$$

Under the condition $Ric(L) \geq K$, by the reversal logarithmic Sobolev inequality due to Bakry and Ledoux [2], for all $t > 0$, we have

$$\frac{|\nabla P_t f|^2}{P_t f} \leq C_K(t) (P_t(f \log f) - P_t f \log P_t f). \quad (40)$$

Hence, for all $K \in \mathbb{R}$, we have

$$\frac{d}{dt} H_K(f, t) \leq 0, \quad \forall t > 0.$$

Taking the time derivative on the both sides of (39), we have

$$\begin{aligned} \frac{d^2}{dt^2} H_K(f, t) &= -C_K(t) D_K(t) \left[\int_M \frac{|\nabla P_t f|^2}{P_t f} + C_K(t) P_t f \log P_t f - K(t) P_t(f \log f) \right] d\mu \\ &\quad + D_K(t) \left[\frac{d}{dt} \int_M \frac{|\nabla P_t f|^2}{P_t f} d\mu - C_K(t) \int_M \frac{|\nabla P_t f|^2}{P_t f} d\mu \right] \\ &\quad + D_K(t) \frac{d}{dt} C_K(t) \int_M (P_t f \log P_t f - P_t(f \log f)) d\mu. \end{aligned}$$

By Bakry and Emery [1] and Li [15, 17], we have

$$\frac{d}{dt} \int_M \frac{|\nabla P_t f|^2}{P_t f} d\mu = -2 \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu - 2 \int_M Ric(L)(\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu.$$

Thus

$$\begin{aligned}
\frac{d^2}{dt^2}H_K(f, t) &= -C_K(t)\frac{d}{dt}H_K(t) - 2D_K(t)\int_M |\nabla^2 \log P_t f|^2 P_t f d\mu \\
&\quad - D_K(t)\int_M (2Ric(L) + C_K(t))(\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu \\
&\quad + D_K(t)\frac{d}{dt}C_K(t)\int_M (P_t f \log P_t f - P_t(f \log f))d\mu. \tag{41}
\end{aligned}$$

Note that, for all $K \in \mathbb{R}$, under the condition $Ric(L) \geq K$, we have

$$2Ric(L) + C_K(t) \geq 2K + \frac{2K}{e^{2Kt} - 1} = \frac{2K}{1 - e^{-2Kt}},$$

and

$$\frac{d}{dt}C_K(t) = \frac{d}{dt} \frac{2K}{e^{2Kt} - 1} = -\frac{2K}{1 - e^{-2Kt}}C_K(t).$$

Substituting these into (41), a simple calculation yields, for all $K \in \mathbb{R}$, and for all $t > 0$,

$$\frac{d^2}{dt^2}H_K(f, t) \leq -2K \coth(Kt) \frac{d}{dt}H_K(t) - 2D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu.$$

Indeed, from (41), we can prove

$$\begin{aligned}
\frac{d^2}{dt^2}H_K(f, t) &= -2K \coth(Kt) \frac{d}{dt}H_K(t) - 2D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu \\
&\quad - 2D_K(t) \int_M (Ric(L) - K)(\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu.
\end{aligned}$$

Let $\alpha_K : (0, \infty) \rightarrow (0, \infty)$ be a C^1 -smooth function. Define the W -entropy by the revised Boltzmann entropy formula

$$W_K(f, t) := \frac{1}{\dot{\alpha}_K(t)} \frac{d}{dt}(\alpha_K(t)H_K(f, t)) = H_K + \frac{\alpha_K}{\dot{\alpha}_K} \dot{H}_K.$$

Set $\beta_K = \frac{\alpha_K}{\dot{\alpha}_K}$. Then

$$\frac{d}{dt}W_K(f, t) = \beta_K(\ddot{H}_K + \frac{1 + \dot{\beta}_K}{\beta_K} \dot{H}_K).$$

Solving the ODE

$$\frac{1 + \dot{\beta}_K}{\beta_K} = 2K \coth(Kt),$$

we can take

$$\beta_K(t) = \frac{\sinh(2Kt)}{2K},$$

and hence

$$\alpha_K(t) = K \tanh(Kt).$$

This yields

$$W_K(f, t) = H_K(f, t) + \frac{\sinh(2Kt)}{2K} \frac{d}{dt}H_K(f, t),$$

and

$$\begin{aligned} \frac{d}{dt}W_K(f, t) &= -\frac{\sinh(2Kt)}{K}D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu \\ &\quad - \frac{\sinh(2Kt)}{K}D_K(t) \int_M (Ric(L) - K)(\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu. \end{aligned}$$

In particular, when $Ric(L) \geq K$, then for all $t > 0$, we have

$$\frac{d}{dt}W_K(f, t) \leq 0.$$

This finishes the proof of Theorem 1.7. \square

3.2 W -entropy for Witten Laplacian with $CD(K, m)$ condition

In this subsection, based on Theorem 2.3, we introduce the W -entropy and prove the W -entropy formula and a rigidity theorem, i.e., Theorem 1.8, for the Witten Laplacian on complete Riemannian manifolds with fixed metrics and potentials satisfying the $CD(K, m)$ condition for $K \in \mathbb{R}$ and $m \in [n, \infty)$.

Let M be a complete Riemannian manifold with bounded geometry condition, $\phi \in C^3(M)$ be such that $\nabla \phi \in C_b^2(M)$. Following [23, 21, 15, 16, 18], we define

$$H_{m,K}(u, t) = - \int_M u \log u d\mu - \Phi_{m,K}(t),$$

where $\Phi_{m,K} \in C((0, \infty), \mathbb{R})$ satisfies

$$\Phi'_{m,K}(t) = \frac{m}{2t} e^{4Kt}, \quad \forall t > 0.$$

Proposition 3.2 *Let M be a complete Riemannian manifold with bounded geometry condition, $\phi \in C^3(M)$ be such that $\nabla \phi \in C_b^2(M)$. Then, under the condition $Ric_{m,n}(L) \geq -K$, we have*

$$\frac{d}{dt}H_{K,m}(u, t) \leq 0.$$

Proof. By the entropy dissipation formula (see [15, 17], and using $\int_M \partial_t u d\mu = \int_M L u d\mu = 0$, we have

$$\frac{d}{dt}H_{m,K}(u, t) = \int_M \left[\frac{|\nabla u|^2}{u^2} - \frac{m}{2t} e^{4Kt} \right] u d\mu \tag{42}$$

$$= \int_M \left[\frac{|\nabla u|^2}{u^2} - \frac{m}{2t} e^{4Kt} - e^{2Kt} \frac{\partial_t u}{u} \right] u d\mu. \tag{43}$$

By the Li-Yau-Hamilton Harnack inequality in Theorem 2.3, we have

$$\frac{d}{dt}H_{m,K}(u, t) \leq 0.$$

\square

Proposition 3.3 *Under the same condition as in Theorem 1.8, we have*

$$\frac{d^2}{dt^2}H_{m,K}(u, t) = -2 \int_M [|\nabla^2 \log u|^2 + Ric(L)(\nabla \log u, \nabla \log u)] u d\mu - \left(\frac{2mK}{t} - \frac{m}{2t^2} \right) e^{4Kt}.$$

Proof. Based on the gradient estimates of the positive solution to the heat equation on complete manifolds with bounded geometry condition (see [15, 17]), we have

$$\frac{d}{dt} \int_M \frac{|\nabla u|^2}{u} d\mu = -2 \int_M [|\nabla^2 \log u|^2 + \text{Ric}(L)(\nabla \log u, \nabla \log u)] u d\mu.$$

Combining this with (42), Proposition 3.3 follows. \square

Following Perelman [23], Ni [21] and [15, 16, 17, 18], we introduce the W -entropy for the heat equation (1) of the Witten Laplacian as follows

$$W_{m,K}(u, t) = \frac{d}{dt} (t H_{m,K}(u, t)).$$

By the entropy dissipation formula, we have

$$\begin{aligned} W_{m,K}(u, t) &= \int_M [t(|\nabla \log u|^2 - \Phi'_{m,K}(t)) - \log u - \Phi_{m,K}(t)] u d\mu \\ &= \int_M [t(2L(-\log u) - |\nabla \log u|^2) - \log u - \Phi_{m,K}(t) - \Phi'_{m,K}(t)] u d\mu. \end{aligned}$$

We are now in a position to state the main result of this section, i.e., Theorem 1.8.

Theorem 3.4 *Let M be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that (M, g) satisfies the bounded geometry condition and $\phi \in C^4(M)$ with $\nabla \phi \in C_b^3(M)$. Then*

$$\begin{aligned} \frac{d}{dt} W_{m,K}(u, t) &= -2t \int_M \left[\left| \nabla^2 \log u + \left(\frac{K}{2} + \frac{1}{2t} \right) g \right|^2 + (\text{Ric}_{m,n}(L) + Kg)(\nabla \log u, \nabla \log u) \right] u d\mu \\ &\quad - \frac{2t}{m-n} \int_M \left| \nabla \phi \cdot \nabla \log u - \frac{(m-n)(1+Kt)}{2t} \right|^2 u d\mu \\ &\quad - \frac{m}{2t} [e^{4Kt}(1+4Kt) - (1+Kt)^2]. \end{aligned}$$

In particular, if $\text{Ric}_{m,n}(L) \geq -K$, then, for all $t \geq 0$, we have

$$\frac{d}{dt} W_{m,K}(u, t) \leq -\frac{m}{2t} [e^{4Kt}(1+4Kt) - (1+Kt)^2].$$

Moreover, the equality holds at some time $t = t_0 > 0$ if and only if M is a quasi-Einstein manifold, i.e., $\text{Ric}_{m,n}(L) = -Kg$, and the potential function $f = -\log u$ satisfies the shrinking soliton equation with respect to $\text{Ric}_{m,n}(L)$, i.e.,

$$\text{Ric}_{m,n}(L) + 2\nabla^2 f = \frac{g}{t},$$

and moreover

$$\nabla \phi \cdot \nabla f = -\frac{(m-n)(1+Kt)}{2t}.$$

Proof. By (42) and Proposition 3.3, we have

$$\begin{aligned} \frac{d}{dt} W_{m,K}(u, t) &= -2t \left[\int_M [|\nabla^2 \log u|^2 + \text{Ric}(L)(\nabla \log u, \nabla \log u)] u d\mu + \left(\frac{mK}{t} - \frac{m}{4t^2} \right) e^{4Kt} \right] \\ &\quad + 2 \int_M \left[\frac{|\nabla u|^2}{u^2} - \frac{m}{2t} e^{4Kt} \right] u d\mu. \end{aligned}$$

Note that

$$\left| \nabla^2 \log u + \left(\frac{e^{2Kt}}{2t} + a(t) \right) g \right|^2 = |\nabla^2 \log u|^2 + 2 \left(\frac{e^{2Kt}}{2t} + a(t) \right) \Delta \log u + n \left(\frac{e^{2Kt}}{2t} + a(t) \right)^2.$$

By direct calculation, we have

$$\begin{aligned} \frac{d}{dt} W_{m,K}(u, t) &= -2t \int_M \left| \nabla^2 \log u + \left(\frac{e^{2Kt}}{2t} + a(t) \right) g \right|^2 u d\mu \\ &\quad - 2t \int_M \left(Ric_{m,n}(L) + \left(2a(t) - \frac{1 - e^{2Kt}}{t} \right) g \right) (\nabla \log u, \nabla \log u) u d\mu \\ &\quad + 2nt \left(\frac{e^{2Kt}}{2t} + a(t) \right)^2 - \frac{me^{4Kt}}{2t} - 2mKe^{4Kt} \\ &\quad + 2(e^{2Kt} + 2ta(t)) \int_M \nabla \phi \cdot \nabla \log u u d\mu - 2t \int_M \frac{|\nabla \phi \cdot \nabla \log u|^2}{m-n} u d\mu. \end{aligned}$$

Let $a(t)$ be chosen such that $2a(t) - \frac{1 - e^{2Kt}}{t} = K$. Then

$$\begin{aligned} \frac{d}{dt} W_{m,K}(u, t) &= -2t \int_M \left[\left| \nabla^2 \log u + \left(\frac{K}{2} + \frac{1}{2t} \right) g \right|^2 + (Ric_{m,n}(L) + Kg)(\nabla \log u, \nabla \log u) \right] u d\mu \\ &\quad + 2nt \left(\frac{1}{2t} + \frac{K}{2} \right)^2 - \frac{me^{4Kt}}{2t} - 2mKe^{4Kt} \\ &\quad + 2(1 + Kt) \int_M \nabla \phi \cdot \nabla \log u u d\mu - 2t \int_M \frac{|\nabla \phi \cdot \nabla \log u|^2}{m-n} u d\mu. \end{aligned}$$

Combining this with

$$\begin{aligned} &\frac{1}{m-n} \int_M \left| \nabla \phi \cdot \nabla \log u - \frac{(m-n)(1+Kt)}{2t} \right|^2 u d\mu \\ &= \frac{(m-n)(1+Kt)^2}{4t^2} - \frac{1+Kt}{t} \int_M \nabla \phi \cdot \nabla \log u u d\mu + \int_M \frac{|\nabla \phi \cdot \nabla \log u|^2}{m-n} u d\mu, \end{aligned}$$

and noting that

$$\begin{aligned} &2nt \left(\frac{1}{2t} + \frac{K}{2} \right)^2 - \frac{me^{4Kt}}{2t} - 2mKe^{4Kt} + \frac{(m-n)(1+Kt)^2}{2t} \\ &= \frac{m}{2t} [(1+Kt)^2 - e^{4Kt}(1+4Kt)], \end{aligned}$$

we can derive the desired W -entropy formula. The rest of the proof is obvious. \square

In particular, taking $m = n$, $\phi \equiv 0$ and g is a fixed Riemannian metric, we obtain the following W -entropy formula for the heat equation of the Laplace-Beltrami operator on Riemannian manifolds, which extends Ni's result in [21] for $K = 0$.

Theorem 3.5 *Let (M, g) be a complete Riemannian manifold with bounded geometry condition. Let u be the fundamental solution to the heat equation $\partial_t u = \Delta u$. Then*

$$\begin{aligned} \frac{d}{dt} W_{n,K}(u, t) &= -2t \int_M \left[\left| \nabla^2 \log u + \left(\frac{K}{2} + \frac{1}{2t} \right) g \right|^2 + (Ric + Kg)(\nabla \log u, \nabla \log u) \right] u d\mu \\ &\quad - \frac{n}{2t} [e^{4Kt}(1+4Kt) - (1+Kt)^2]. \end{aligned}$$

In particular, if $\text{Ric} \geq -K$, then, for all $t \geq 0$, we have

$$\frac{d}{dt}W_{n,K}(u, t) \leq -\frac{n}{2t} [e^{4Kt}(1 + 4Kt) - (1 + Kt)^2].$$

Moreover, the equality holds at some time $t = t_0 > 0$ if and only if M is an Einstein manifold, i.e., $\text{Ric} = -Kg$, and the potential function $f = -\log u$ satisfies the shrinking soliton equation, i.e.,

$$\text{Ric} + 2\nabla^2 f = \frac{g}{t}.$$

To end this subsection, let us remark that in our previous paper [18] we introduced another W -entropy functional for the heat equation associated with the Witten Laplacian satisfying the $CD(K, m)$ -condition as follows

$$\widetilde{W}_{m,K}(u) = \frac{d}{dt}(t\widetilde{H}_{m,K}(u)),$$

where

$$\widetilde{H}_{m,K}(u) = -\int_M u \log u d\mu - \frac{m}{2t}(1 + \log(4\pi t)) - \frac{mKt}{2}(1 + \frac{1}{6}Kt),$$

and we proved that

$$\begin{aligned} \frac{d}{dt}\widetilde{W}_{m,K}(u) &= -2t \int_M \left[\left| \nabla^2 \log u + \left(\frac{K}{2} + \frac{1}{2t} \right) g \right|^2 + (\text{Ric}_{m,n}(L) + Kg)(\nabla \log u, \nabla \log u) \right] u d\mu \\ &\quad - \frac{2t}{m-n} \int_M \left| \nabla \phi \cdot \nabla \log u - \frac{(m-n)(1+Kt)}{2t} \right|^2 u d\mu. \end{aligned}$$

Indeed, letting $\Psi_{m,K}(t) = \Phi_{m,K}(t) - \frac{m}{2t}(1 + \log(4\pi t)) - \frac{mKt}{2}(1 + \frac{1}{6}Kt)$, we have

$$\widetilde{W}_{m,K}(u) - W_{m,K}(u) = \frac{d}{dt}(t\Psi_{m,K}(t)),$$

and

$$\frac{d}{dt}(\widetilde{W}_{m,K}(u) - W_{m,K}(u)) = \frac{d^2}{dt^2}(t\Psi_{m,K}(t)) = \frac{m}{2t} [e^{4Kt}(1 + 4Kt) - (1 + Kt)^2].$$

3.3 W -entropy for Witten Laplacian on K -super Perelman Ricci flow

In this subsection, we extend the W -entropy formula to the time dependent Witten Laplacian on compact Riemannian manifolds with K -super Perelman Ricci flow.

Let $(M, g(t), \phi(t), t \in [0, T])$ be a complete Riemannian manifold with a family of time dependent metrics $g(t)$ and potentials $\phi(t)$. Let

$$L = \Delta_{g(t)} - \nabla_{g(t)}\phi(t) \cdot \nabla_{g(t)}$$

be the time dependent Witten Laplacian on $(M, g(t), \phi(t))$. Let

$$d\mu(t) = e^{-\phi(t)} d\text{vol}_{g(t)}.$$

Suppose that

$$\frac{\partial \phi}{\partial t} = \frac{1}{2} \text{Tr} \frac{\partial g}{\partial t}. \tag{44}$$

Then $\mu(t)$ is indeed independent of $t \in [0, T]$, i.e.,

$$\frac{\partial \mu(t)}{\partial t} = 0, \quad t \in [0, T].$$

We now state the main results of this subsection, which extend Theorem 1.7 and Theorem 1.8 to the time dependent Witten Laplacian on compact Riemannian manifolds with K -super Perelman Ricci flow.

Theorem 3.6 *Let $(M, g(t), t \in [0, T])$ be a compact Riemannian manifold with a family of metrics $g(t)$, and $\phi \in C^{2,1}(M \times [0, T])$. Suppose that (44) holds and*

$$\frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) \geq K,$$

where $K \in \mathbb{R}$ is a constant. Let $u(\cdot, t) = P_t f$ be a positive solution to the heat equation $\partial_t u = Lu$ with $u(\cdot, 0) = f$, f is a positive and measurable function on M . Define

$$H_K(f, t) = D_K(t) \int_M (P_t(f \log f) - P_t f \log P_t f) d\mu,$$

where $D_0(t) = \frac{1}{t}$ and $D_K(t) = \frac{1}{|1 - e^{-2Kt}|}$ for $K \neq 0$. Then, for all $K \in \mathbb{R}$,

$$\frac{d}{dt} H_K(f, t) \leq 0, \quad \forall t \in (0, T],$$

and for all $K \in \mathbb{R}$ and $t \in (0, T]$, we have

$$\frac{d^2}{dt^2} H_K(t) + 2K \coth(Kt) \frac{d}{dt} H_K(t) \leq -2D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu.$$

Define the W -entropy by the revised Boltzmann entropy formula

$$W_K(f, t) = H_K(f, t) + \frac{\sinh(2Kt)}{2K} \frac{d}{dt} H_K(f, t).$$

Then, for all $K \in \mathbb{R}$, and for all $t \in (0, T]$, we have

$$\begin{aligned} \frac{d}{dt} W_K(f, t) &= -\frac{\sinh(2Kt)}{K} D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu \\ &\quad - \frac{\sinh(2Kt)}{K} D_K(t) \int_M \left(\frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) - K \right) (\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu. \end{aligned}$$

In particular, for all $K \in \mathbb{R}$, we have

$$\frac{d}{dt} W_K(f, t) \leq 0, \quad \forall t \in (0, T].$$

Proof. By Li-Li [18], we have

$$\frac{d}{dt} \int_M \frac{|\nabla P_t f|^2}{P_t f} d\mu = -2 \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu - 2 \int_M \left(\frac{1}{2} \frac{\partial g}{\partial t} + Ric(L) \right) (\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu.$$

The rest of the proof is similar the one of Theorem 1.7. \square

Theorem 3.7 *Let $(M, g(t), \phi(t), t \in [0, T])$ be a compact Riemannian manifold with a family of time dependent metrics and potentials $(g(t), \phi(t), t \in [0, T])$ satisfying (44). Let $m \geq n$ and $K \geq 0$ be two constants which are independent of $t \in [0, T]$. Let u be a positive solution to the heat equation $\partial_t u = Lu$. Let*

$$W_{m,K}(u, t) = \frac{d}{dt}(tH_{m,K}(u, t)),$$

where

$$H_{m,K}(u, t) = - \int_M u \log u d\mu - \Phi_{m,K}(t).$$

Then, for all $t \in [0, T]$, we have

$$\begin{aligned} \frac{d}{dt}W_{m,K}(u, t) &= -2t \int_M \left| \nabla^2 \log u + \left(\frac{K}{2} + \frac{1}{2t} \right) g \right|^2 u d\mu - \frac{m}{2t} [e^{4Kt}(1 + 4Kt) - (1 + Kt)^2] \\ &\quad - 2t \int_M \left(\frac{1}{2} \frac{\partial g}{\partial t} + Ric_{m,n}(L) + Kg \right) (\nabla \log u, \nabla \log u) u d\mu \\ &\quad - \frac{2t}{m-n} \int_M \left| \nabla \phi \cdot \nabla \log u - \frac{(m-n)(1+Kt)}{2t} \right|^2 u d\mu. \end{aligned}$$

In particular, if $(M, g(t), \phi(t), t \in [0, T])$ is a K -super Perelman Ricci flow with respect to the m -dimensional Bakry-Emery Ricci curvature $Ric_{m,n}(L)$,

$$\frac{1}{2} \frac{\partial g}{\partial t} + Ric_{m,n}(L) \geq -Kg,$$

we have

$$\frac{d}{dt}W_{m,K}(u, t) \leq -\frac{m}{2t} [e^{4Kt}(1 + 4Kt) - (1 + Kt)^2].$$

Moreover, the equality holds at some time $t = t_0 > 0$ if and only if $(M, g(t), t \in [0, T])$ is a quasi-Ricci flow, i.e.,

$$\begin{aligned} \frac{1}{2} \frac{\partial g}{\partial t} &= -Ric_{m,n}(L) - Kg, \\ \frac{\partial \phi}{\partial t} &= -R - \Delta \phi + \frac{|\nabla \phi|^2}{m-n} - nK, \end{aligned}$$

and the potential function $f = -\log u$ satisfies

$$\begin{aligned} 2\nabla^2 f &= \left(\frac{1}{t} + K \right) g, \\ \nabla \phi \cdot \nabla f &= -\frac{(m-n)(1+Kt)}{2t}. \end{aligned}$$

Proof. The proof is similar to the one of Theorem 1.8. See [18] for the case $K = 0$. \square

Similarly to the end of Section 3.2, we can reformulate Theorem 3.7 in terms of $\widetilde{W}_{m,K}$. See [18].

4 The Li-Yau and the Li-Yau-Hamilton Harnack inequalities on compact super-Ricci flows

In this section we prove the Li-Yau Harnack inequality and the Li-Yau-Hamilton Harnack inequality on compact Riemannian manifolds equipped with variants of the (K, m) -super Ricci flow. In the literature, the Li-Yau Harnack inequality for heat equation $\partial_t u = \Delta u$ on compact Ricci flow has been studied by many authors. See [4, 6, 27] and references therein.

4.1 The commutator $[\partial_t, L]f$

Let M be a compact manifold with a family of time dependent metrics $(g(t), t \in [0, T])$ and potentials $\phi(t) \in C^2(M)$, $t \in [0, T]$. Let $\partial_t g = 2h$.

Lemma 4.1 *For any $f \in C^\infty(M)$, it holds*

$$\partial_t |\nabla f|^2 = -2h(\nabla f, \nabla f) + 2\langle \nabla f, \nabla f_t \rangle,$$

and

$$[\partial_t, L]f = -2\langle h, \nabla^2 f \rangle + 2h(\nabla \phi, \nabla f) - \langle 2\operatorname{div} h - \nabla \operatorname{Tr}_g h + \nabla \partial_t \phi, \nabla f \rangle.$$

Proof. By direct calculation, cf. [5, 27], we have

$$\partial_t \Delta_{g(t)} f = \Delta_{g(t)} \partial_t f - 2\langle h, \nabla^2 f \rangle - 2\langle \operatorname{div} h - \frac{1}{2} \nabla \operatorname{Tr}_g h, \nabla f \rangle,$$

and

$$\partial_t \langle \nabla \phi, \nabla f \rangle = -\partial_t g(\nabla \phi, \nabla f) + \langle \nabla \phi_t, \nabla f \rangle + \langle \nabla \phi, \nabla f_t \rangle.$$

Therefore

$$\begin{aligned} \partial_t Lf &= \partial_t \Delta_{g(t)} f - \partial_t \langle \nabla \phi, \nabla f \rangle \\ &= \Delta_{g(t)} \partial_t f - 2\langle h, \nabla^2 f \rangle - 2\langle \operatorname{div} h - \frac{1}{2} \nabla \operatorname{Tr}_g h, \nabla f \rangle \\ &\quad + 2h(\nabla \phi, \nabla f) - \langle \nabla \phi_t, \nabla f \rangle - \langle \nabla \phi, \nabla f_t \rangle \\ &= L\partial_t f - 2\langle h, \nabla^2 f \rangle + 2h(\nabla \phi, \nabla f) - \langle 2\operatorname{div} h - \nabla \operatorname{Tr}_g h + \nabla \phi_t, \nabla f \rangle. \end{aligned}$$

This finishes the proof. \square

4.2 The Li-Yau Harnack inequality

Let u be a positive solution to the heat equation $\partial_t u = Lu$. Let $f = \log u$. Then

$$(L - \partial_t)f = -|\nabla f|^2.$$

Let

$$F = t(|\nabla f|^2 - \alpha f_t).$$

We have

Lemma 4.2

$$(L - \partial_t)F = 2t(|\nabla^2 f|^2 + \operatorname{Ric}(L)(\nabla f, \nabla f) + (1 - \alpha)h(\nabla f, \nabla f)) - 2\langle \nabla f, \nabla F \rangle - t^{-1}F + \alpha t[\partial_t, L]f.$$

Proof. By the Bochner formula and using

$$\partial_t |\nabla f|_{g(t)}^2 = -\partial_t g(t)(\nabla f, \nabla f) + 2\langle \nabla f, \nabla f_t \rangle_{g(t)},$$

we have

$$\begin{aligned} LF &= tL|\nabla f|^2 - \alpha tLf_t \\ &= 2t(|\nabla^2 f|^2 + \operatorname{Ric}(L)(\nabla f, \nabla f) + \langle \nabla f, \nabla Lf \rangle) - \alpha tL\partial_t f \\ &= 2t(|\nabla^2 f|^2 + \operatorname{Ric}(L)(\nabla f, \nabla f) + \langle \nabla f, \nabla(f_t - |\nabla f|^2) \rangle) - \alpha tL\partial_t f \\ &= 2t(|\nabla^2 f|^2 + \operatorname{Ric}(L)(\nabla f, \nabla f)) - 2\langle \nabla f, \nabla F \rangle + 2(1 - \alpha)t\langle \nabla f, \nabla f_t \rangle - \alpha tL\partial_t f \\ &= 2t(|\nabla^2 f|^2 + \operatorname{Ric}(L)(\nabla f, \nabla f)) - 2\langle \nabla f, \nabla F \rangle \\ &\quad + 2t(1 - \alpha)h(\nabla f, \nabla f) + (1 - \alpha)t\partial_t |\nabla f|^2 - \alpha tL\partial_t f. \end{aligned}$$

On the other hand

$$\begin{aligned}
\partial_t F &= (|\nabla f|^2 - \alpha f_t) + t\partial_t |\nabla f|^2 - \alpha t f_{tt} \\
&= (|\nabla f|^2 - \alpha f_t) + t\partial_t |\nabla f|^2 - \alpha t \partial_t (Lf + |\nabla f|^2) \\
&= (|\nabla f|^2 - \alpha f_t) + (1 - \alpha)t\partial_t |\nabla f|^2 - \alpha t \partial_t Lf.
\end{aligned}$$

Thus

$$(L - \partial_t)F = 2t(|\nabla^2 f|^2 + (Ric(L) + (1 - \alpha)h)(\nabla f, \nabla f)) - 2\langle \nabla f, \nabla F \rangle - t^{-1}F + \alpha t[\partial_t, L]f.$$

By Lemma 4.1, it holds

$$[\partial_t, L]f = -2\langle h, \nabla^2 f \rangle + 2h(\nabla \phi, \nabla f) - \langle 2\operatorname{div} h - \nabla \operatorname{Tr}_g h + \nabla \phi_t, \nabla f \rangle.$$

Thus

$$\begin{aligned}
(L - \partial_t)F &= 2t \left| \nabla^2 f - \frac{\alpha h}{2} \right|^2 - \frac{t\alpha^2 |h|^2}{2} + 2t(Ric(L) + (1 - \alpha)h)(\nabla f, \nabla f) \\
&\quad - 2\langle \nabla f, \nabla F \rangle - t^{-1}F + \alpha t S_1(\nabla f),
\end{aligned}$$

where

$$S_1(\nabla f) = 2h(\nabla \phi, \nabla f) - \langle 2\operatorname{div} h - \nabla \operatorname{Tr}_g h + \nabla \phi_t, \nabla f \rangle.$$

Using the Cauchy-Schwartz inequality, for all $\varepsilon > 0$, we have $(a + b)^2 \geq \frac{a^2}{1+\varepsilon} - \frac{b^2}{\varepsilon}$. Thus

$$\begin{aligned}
\left| \nabla^2 f - \frac{\alpha h}{2} \right|^2 &\geq \frac{1}{n} \left| \Delta f - \frac{\alpha \operatorname{Tr} h}{2} \right|^2 \\
&\geq \frac{|Lf|^2}{n(1+\varepsilon)} - \frac{|\nabla \phi \cdot \nabla f - \frac{\alpha \operatorname{Tr} h}{2}|^2}{n\varepsilon}.
\end{aligned}$$

Let $m := n(1 + \varepsilon)$. Then

$$\begin{aligned}
(L - \partial_t)F &\geq \frac{2t}{m} |Lf|^2 - \frac{2t}{m-n} \left| \nabla \phi \cdot \nabla f - \frac{\alpha \operatorname{Tr} h}{2} \right|^2 - \frac{t\alpha^2 |h|^2}{2} + 2t(Ric(L) + (1 - \alpha)h)(\nabla f, \nabla f) \\
&\quad - 2\langle \nabla f, \nabla F \rangle - t^{-1}F + \alpha t S_1(\nabla f) \\
&= \frac{2t}{m} |Lf|^2 - \frac{t\alpha^2 (\operatorname{Tr} h)^2}{2(m-n)} - \frac{t\alpha^2 |h|^2}{2} + 2t(Ric_{m,n}(L) + (1 - \alpha)h)(\nabla f, \nabla f) \\
&\quad - 2\langle \nabla f, \nabla F \rangle - t^{-1}F + \alpha t S_1(\nabla f) + \frac{2\alpha t \operatorname{Tr} h}{m-n} \langle \nabla \phi, \nabla f \rangle. \tag{45}
\end{aligned}$$

Let

$$S(\cdot) = S_1(\cdot) + \frac{2\operatorname{Tr} h}{m-n} \langle \nabla \phi, \cdot \rangle.$$

Substituting $Lf = |\nabla f|^2 - f_t = \frac{F}{\alpha t} + \frac{\alpha-1}{\alpha} |\nabla f|^2$ into (45), we have

$$\begin{aligned}
(L - \partial_t)F &\geq \frac{2t}{m} \left[\frac{F}{\alpha t} + \frac{\alpha-1}{\alpha} |\nabla f|^2 \right]^2 - \frac{t\alpha^2}{2} \left[\frac{(\operatorname{Tr} h)^2}{m-n} + |h|^2 \right] \\
&\quad + 2t(Ric_{m,n}(L) + (1 - \alpha)h)(\nabla f, \nabla f) - 2\langle \nabla f, \nabla F \rangle - t^{-1}F + \alpha t S(\nabla f).
\end{aligned}$$

Hence, as $F \geq 0$, and $\alpha \geq 1$, we have

$$\begin{aligned} (L - \partial_t)F &\geq \frac{2F^2}{\alpha^2 mt} + \frac{4(\alpha - 1)}{m\alpha^2} |\nabla f|^2 F + \frac{2t(\alpha - 1)^2}{m\alpha^2} |\nabla f|^4 - \frac{t\alpha^2}{2} \left[\frac{(\text{Tr}h)^2}{m - n} + |h|^2 \right] \\ &\quad + 2t(\text{Ric}_{m,n}(L) + (1 - \alpha)h)(\nabla f, \nabla f) - 2\langle \nabla f, \nabla F \rangle - t^{-1}F + \alpha t S(\nabla f). \end{aligned} \quad (46)$$

Under the assumption

$$\text{Ric}_{m,n}(L) + (1 - \alpha)h \geq -K, \quad (47)$$

and setting

$$A^2 = \max \left[|h|^2 + \frac{(\text{Tr}h)^2}{m - n} \right], \quad B = \max |S|,$$

we have

$$\begin{aligned} (L - \partial_t)F &\geq \frac{2F^2}{\alpha^2 mt} + \frac{2t(\alpha - 1)^2}{m\alpha^2} |\nabla f|^4 - \frac{t\alpha^2 A^2}{2} \\ &\quad - 2Kt |\nabla f|^2 - 2\langle \nabla f, \nabla F \rangle - t^{-1}F - \alpha Bt |\nabla f|. \end{aligned}$$

Using the inequality

$$ax^4 + bx^2 + cx \geq -\frac{(b - \gamma)^2}{4a} - \frac{c^2}{4\gamma}, \quad (48)$$

where $\gamma > 0$ is any positive constant, we can derive that

$$\frac{2t(\alpha - 1)^2}{m\alpha^2} |\nabla f|^4 - 2Kt |\nabla f|^2 - \alpha Bt |\nabla f| \geq -\frac{m\alpha^2 t(2K + \gamma)^2}{8(\alpha - 1)^2} - \frac{\alpha^2 B^2 t}{4\gamma}. \quad (49)$$

Hence

$$\begin{aligned} (L - \partial_t)F &\geq \frac{2F^2}{\alpha^2 mt} - \frac{F}{t} - 2\langle \nabla f, \nabla F \rangle - \frac{t\alpha^2 A^2}{2} \\ &\quad - \frac{m\alpha^2 t(2K + \gamma)^2}{8(\alpha - 1)^2} - \frac{\alpha^2 B^2 t}{4\gamma}. \end{aligned}$$

Let (x_0, t_0) be the point where F achieves the maximum on $M \times [0, T]$. Then $\nabla F(x_0, t_0) = 0$, $\Delta F(x_0, t_0) \leq 0$ and $\partial_t F(x_0, t_0) \geq 0$. Therefore, at (x_0, t_0) ,

$$(L - \partial_t)F \leq 0,$$

i.e.,

$$0 \geq \frac{2F^2}{\alpha^2 mt_0} - \frac{F}{t_0} - \frac{t_0 \alpha^2 A^2}{2} - \frac{m\alpha^2 t_0(2K + \gamma)^2}{8(\alpha - 1)^2} - \frac{\alpha^2 B^2 t_0}{4\gamma}. \quad (50)$$

Thus

$$F \leq \frac{m\alpha^2}{4} \left[1 + \sqrt{1 + \frac{t_0^2}{m} \left(4A^2 + \frac{(2K + \gamma)^2}{(\alpha - 1)^2} + \frac{2B^2}{\gamma} \right)} \right]. \quad (51)$$

Note that when $B = 0$, we can take $\gamma = 0$ in (48), (50) and (51), i.e.,

$$F \leq \frac{m\alpha^2}{4} \left[1 + \sqrt{1 + \frac{t_0^2}{m} \left(4A^2 + \frac{4K^2}{(\alpha - 1)^2} \right)} \right]. \quad (52)$$

and if $K = 0$, i.e., if

$$Ric_{m,n}(L) + (1 - \alpha)h \geq 0, \quad (53)$$

we have

$$F \leq \frac{m\alpha^2}{4} \left[1 + \sqrt{1 + \frac{4T^2 A^2}{m}} \right]. \quad (54)$$

In particular, when $A = B = 0$, and $Ric_{m,n}(L) \geq 0$, we can take $\alpha \rightarrow 1$ and recapture the Li-Yau Harnack inequality [12]

$$|\nabla f|^2 - f_t \leq \frac{m}{2t}. \quad (55)$$

Therefore, we have proved the following Li-Yau Harnack inequality for positive solutions to the heat equation $\partial_t u = Lu$ on compact Riemannian manifolds equipped with the backward (K, m) -super Ricci flows.

Theorem 4.3 *Let $(M, g(t), t \in [0, T])$ be a compact Riemannian manifold with a family of time dependent metrics $g(t)$ and potentials $\phi(t) \in C^2(M)$, $t \in [0, T]$. Let u be a positive solution to the heat equation $\partial_t u = Lu$. Let $\partial_t g = 2h$ and $\alpha > 1$. Suppose that*

$$\frac{1}{2}(1 - \alpha)\partial_t g + Ric_{m,n}(L) \geq -Kg, \quad (56)$$

and assuming that $A^2 = \max \left[|h|^2 + \frac{(\text{Tr}h)^2}{m-n} \right] < \infty$ and $B = \max |S| < \infty$, where

$$S(\cdot) = 2h(\nabla\phi, \cdot) - \langle 2\text{div}h - \nabla\text{Tr}h + \nabla\phi_t, \cdot \rangle + \frac{2\text{Tr}h}{m-n} \langle \nabla\phi, \cdot \rangle.$$

Then for any $\gamma > 0$ and for all $t \in (0, T]$, we have

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{\partial_t u}{u} \leq \frac{m\alpha^2}{4t} \left[1 + \sqrt{1 + \frac{T^2}{m} \left(4A^2 + \frac{(2K + \gamma)^2}{(\alpha - 1)^2} + \frac{2B^2}{\gamma} \right)} \right].$$

In the case $B = 0$, for all $t \in (0, T]$, we have

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{\partial_t u}{u} \leq \frac{m\alpha^2}{4t} \left[1 + \sqrt{1 + \frac{T^2}{m} \left(4A^2 + \frac{4K^2}{(\alpha - 1)^2} \right)} \right].$$

In particular, in the case $A = B = 0$ and $Ric_{m,n}(L) \geq 0$, we have the Li-Yau Harnack inequality

$$\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{m}{2t}.$$

4.3 The Li-Yau-Hamilton Harnack inequality

Let u be a positive solution to the heat equation $\partial_t u = Lu$. Let $f = \log u$. Then $(\partial_t - L)f = |\nabla f|^2$. Let

$$F = te^{-2Kt}(e^{-2Kt}|\nabla f|^2 - f_t) = te^{-4Kt}|\nabla f|^2 - te^{-2Kt}f_t.$$

In this section we prove the following Li-Yau-Hamilton Harnack inequality on a variant of the (K, m) -super Ricci flow on compact manifolds.

Theorem 4.4 *Let $(M, g(t), t \in [0, T])$ be a compact Riemannian manifold with a family of time dependent metrics $g(t)$ and potentials $\phi(t) \in C^2(M)$, $t \in [0, T]$. Let u be a positive solution to the heat equation $\partial_t u = Lu$. Suppose that $\partial_t g = 2h$ satisfies*

$$e^{-4Kt}(h + \text{Ric}_{m,n}(L) + Kg) - e^{-2Kt}h \geq \alpha_K(t)g, \quad (57)$$

and

$$A^2 = \max \left[|h|^2 + \frac{(\text{Tr}h)^2}{m-n} \right] < \infty, \quad B = \max |S| < \infty,$$

where

$$S(\cdot) = \left\langle \frac{2\text{Tr}h}{m-n} \nabla \phi - 2\text{div}h + \nabla \text{Tr}_g h - \nabla \partial_t \phi, \cdot \right\rangle + 2h(\nabla \phi, \cdot).$$

Then, for any $\gamma > 0$ and $t \in [0, T]$, we have

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt} \frac{\partial_t u}{u} \leq \frac{me^{4Kt}}{2t} \left[1 + \sqrt{\frac{A^2 T^2}{m} + \max_{t \in [0, T]} \frac{t^2 (2\alpha_K(t) - \gamma)^2}{4e^{-4Kt}(1 - e^{-2Kt})^2} + \max_{t \in [0, T]} \frac{t^2 e^{-4Kt} B^2}{2m\gamma}} \right].$$

In the case $B = 0$, we have

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt} \frac{\partial_t u}{u} \leq \frac{me^{4Kt}}{2t} \left[1 + \sqrt{\frac{A^2 T^2}{m} + \max_{t \in [0, T]} \frac{t^2 \alpha_K^2(t)}{e^{-4Kt}(1 - e^{-2Kt})^2}} \right].$$

and if $\alpha_K(t) = 0$, i.e., if

$$e^{-4Kt}(h + \text{Ric}_{m,n}(L) + K) - e^{-2Kt}h \geq 0, \quad (58)$$

we have

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt} \frac{\partial_t u}{u} \leq \frac{me^{4Kt}}{2t} \left[1 + \frac{TA}{\sqrt{m}} \right].$$

In particular, when $A = B = 0$, and $\text{Ric}_{m,n}(L) \geq -K$, we recapture Hamilton's Harnack inequality [10]

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt} \frac{\partial_t u}{u} \leq \frac{me^{4Kt}}{2t}.$$

Proof. If $F \leq 0$ on $[0, T] \times M$, we have $\frac{|\nabla u|^2}{u^2} - e^{2Kt} \frac{\partial_t u}{u} \leq 0$. In this case, the Li-Yau-Hamilton Harnack inequality automatically holds. Thus, in order to prove Theorem 4.4, we need only to consider the case $F \geq 0$. Notice that

$$\begin{aligned} L|\nabla f|^2 &= 2|Hess f|^2 + 2\langle \nabla f, \nabla Lf \rangle + 2\text{Ric}(L)(\nabla f, \nabla f), \\ \partial_t |\nabla f|^2 &= 2\langle \nabla \partial_t f, \nabla f \rangle - 2h(\nabla f, \nabla f), \\ \langle \nabla F, \nabla f \rangle &= 2te^{-4Kt}(1 - e^{2Kt})\nabla^2 f(\nabla f, \nabla f) - te^{-2Kt}\langle \nabla Lf, \nabla f \rangle \\ &= te^{-2Kt}[2(e^{-2Kt} - 1)\nabla^2 f(\nabla f, \nabla f) - \langle \nabla Lf, \nabla f \rangle]. \end{aligned}$$

Then

$$\begin{aligned}
(\partial_t - L)F &= \partial_t(te^{-2Kt}(e^{-2Kt}|\nabla f|^2 - f_t)) - te^{-2Kt}(e^{-2Kt}L|\nabla f|^2 - Lf_t) \\
&= e^{-2Kt}(1 - 2Kt)(e^{-2Kt}|\nabla f|^2 - f_t) + te^{-2Kt}[e^{-2Kt}\partial_t|\nabla f|^2 - 2Ke^{-2Kt}|\nabla f|^2 - f_{tt}) \\
&\quad - te^{-2Kt}(e^{-2Kt}L|\nabla f|^2 - Lf_t) \\
&= \frac{1 - 2Kt}{t}F + te^{-2Kt}[e^{-2Kt}(\partial_t - L)|\nabla f|^2 - 2Ke^{-2Kt}|\nabla f|^2 - (\partial_t - L)f_t] \\
&= \frac{1 - 2Kt}{t}F + te^{-2Kt}[e^{-2Kt}(\partial_t - L)|\nabla f|^2 - 2Ke^{-2Kt}|\nabla f|^2 - \partial_t|\nabla f|^2 - [\partial_t, L]f] \\
&= \frac{1 - 2Kt}{t}F + te^{-2Kt}[(e^{-2Kt} - 1)\partial_t|\nabla f|^2 - e^{-2Kt}L|\nabla f|^2 - 2Ke^{-2Kt}|\nabla f|^2 - [\partial_t, L]f] \\
&= \frac{1 - 2Kt}{t}F + te^{-2Kt}[(e^{-2Kt} - 1)(2\langle \nabla \partial_t f, \nabla f \rangle - 2h(\nabla f, \nabla f)) \\
&\quad - e^{-2Kt}(2|Hess f|^2 + 2\langle \nabla f, \nabla Lf \rangle + 2Ric(L)(\nabla f, \nabla f)) - 2Ke^{-2Kt}|\nabla f|^2 - [\partial_t, L]f] \\
&= \frac{1 - 2Kt}{t}F + te^{-2Kt}[(e^{-2Kt} - 1)(2\langle \nabla Lf, \nabla f \rangle + 2\langle \nabla|\nabla f|^2, \nabla f \rangle - 2h(\nabla f, \nabla f)) \\
&\quad - e^{-2Kt}(2|Hess f|^2 + 2\langle \nabla f, \nabla Lf \rangle + 2Ric(L)(\nabla f, \nabla f)) - 2Ke^{-2Kt}|\nabla f|^2 - [\partial_t, L]f] \\
&= \frac{1 - 2Kt}{t}F + te^{-2Kt}[(e^{-2Kt} - 1)(4\nabla^2 f(\nabla f, \nabla f) - 2h(\nabla f, \nabla f)) - 2\langle \nabla f, \nabla Lf \rangle \\
&\quad - e^{-2Kt}(2|Hess f|^2 + 2(Ric(L) + K)(\nabla f, \nabla f)) - [\partial_t, L]f] \\
&= \frac{1 - 2Kt}{t}F + 2\langle \nabla F, \nabla f \rangle + te^{-2Kt}[-e^{-2Kt}(2|Hess f|^2 + 2(h + Ric(L) + K)(\nabla f, \nabla f)) \\
&\quad + 2h(\nabla f, \nabla f) - [\partial_t, L]f].
\end{aligned}$$

This yields

$$\begin{aligned}
(L - \partial_t)F &= \frac{2Kt - 1}{t}F - 2\langle \nabla F, \nabla f \rangle + te^{-2Kt}[e^{-2Kt}(2|Hess f|^2 + 2(h + Ric(L) + K)(\nabla f, \nabla f)) \\
&\quad - 2h(\nabla f, \nabla f) - 2\langle h, \nabla^2 f \rangle + 2h(\nabla \phi, \nabla f) - \langle 2\text{div} h - \nabla \text{Tr}_g h + \nabla \partial_t \phi, \nabla f \rangle] \\
&= \frac{2Kt - 1}{t}F - 2\langle \nabla F, \nabla f \rangle + 2te^{-4Kt} \left| \nabla^2 f - \frac{e^{2Kt}}{2}h \right|^2 - \frac{t}{2}|h|^2 \\
&\quad + 2te^{-4Kt}(h + Ric(L) + K)(\nabla f, \nabla f) - 2te^{-2Kt}h(\nabla f, \nabla f) \\
&\quad + te^{-2Kt}(2h(\nabla \phi, \nabla f) - \langle 2\text{div} h - \nabla \text{Tr}_g h + \nabla \partial_t \phi, \nabla f \rangle).
\end{aligned}$$

Note that

$$\left| \nabla^2 f - \frac{e^{2Kt}}{2}h \right|^2 \geq \frac{1}{n} \left| \Delta f - \frac{e^{2Kt}}{2}\text{Tr} h \right|^2$$

Using the elementary inequality $(a + b)^2 \geq \frac{a^2}{1 + \varepsilon} - \frac{b^2}{\varepsilon}$ with $a = Lf$, $b = \nabla \phi \cdot \nabla f - \frac{e^{2Kt}}{2}\text{Tr} h$, we have

$$\left| \Delta f - \frac{e^{2Kt}}{2}\text{Tr} h \right|^2 \geq \frac{|Lf|^2}{1 + \varepsilon} - \frac{1}{\varepsilon} \left| \nabla \phi \cdot \nabla f - \frac{e^{2Kt}}{2}\text{Tr} h \right|^2$$

Taking $\varepsilon = \frac{m-n}{n}$, we obtain

$$\left| \nabla^2 f - \frac{e^{2Kt}}{2}h \right|^2 \geq \frac{|Lf|^2}{m} - \frac{1}{m-n} \left| \nabla \phi \cdot \nabla f - \frac{e^{2Kt}}{2}\text{Tr} h \right|^2.$$

This yields

$$\begin{aligned}
(L - \partial_t)F &\geq \frac{2Kt-1}{t}F - 2\langle \nabla F, \nabla f \rangle + \frac{2te^{-4Kt}}{m}|Lf|^2 - \frac{2te^{-4Kt}}{m-n} \left| \nabla \phi \cdot \nabla f - \frac{e^{2Kt}}{2} \text{Tr} h \right|^2 \\
&\quad + 2te^{-4Kt}(h + \text{Ric}(L) + K)(\nabla f, \nabla f) - 2te^{-2Kt}h(\nabla f, \nabla f) - \frac{t}{2}|h|^2 \\
&\quad + te^{-2Kt}(2h(\nabla \phi, \nabla f) - \langle 2\text{div} h - \nabla \text{Tr}_g h + \nabla \partial_t \phi, \nabla f \rangle) \\
&= \frac{2Kt-1}{t}F - 2\langle \nabla F, \nabla f \rangle + \frac{2te^{-4Kt}}{m}|Lf|^2 - \frac{t}{2} \left[\frac{(\text{Tr} h)^2}{m-n} + |h|^2 \right] \\
&\quad + 2te^{-4Kt}(h + \text{Ric}_{m,n}(L) + K)(\nabla f, \nabla f) - 2te^{-2Kt}h(\nabla f, \nabla f) \\
&\quad + te^{-2Kt} \left(\frac{2\text{Tr} h}{m-n} \langle \nabla \phi, \nabla f \rangle + 2h(\nabla \phi, \nabla f) - \langle 2\text{div} h - \nabla \text{Tr}_g h + \nabla \partial_t \phi, \nabla f \rangle \right).
\end{aligned}$$

Substituting $Lf = (e^{-2Kt} - 1)|\nabla f|^2 - \frac{e^{2Kt}}{t}F$ into the above inequality, we get

$$\begin{aligned}
(L - \partial_t)F &\geq \frac{2Kt-1}{t}F - 2\langle \nabla F, \nabla f \rangle - \frac{t}{2} \left[\frac{(\text{Tr} h)^2}{m-n} + |h|^2 \right] \\
&\quad + \frac{2te^{-4Kt}}{m} \left| (e^{-2Kt} - 1)|\nabla f|^2 - \frac{e^{2Kt}}{t}F \right|^2 \\
&\quad + 2te^{-4Kt}(h + \text{Ric}_{m,n}(L) + K)(\nabla f, \nabla f) - 2te^{-2Kt}h(\nabla f, \nabla f) \\
&\quad + te^{-2Kt} \left(\frac{2\text{Tr} h}{m-n} \langle \nabla \phi, \nabla f \rangle + 2h(\nabla \phi, \nabla f) - \langle 2\text{div} h - \nabla \text{Tr}_g h + \nabla \partial_t \phi, \nabla f \rangle \right).
\end{aligned}$$

Under the assumption

$$e^{-4Kt}(h + \text{Ric}_{m,n}(L) + K) - e^{-2Kt}h \geq \alpha_K(t), \quad (59)$$

where $\alpha_K(t)$ is a function of t and K , we have (using the assumption $F \geq 0$ and $K \geq 0$)

$$\begin{aligned}
(L - \partial_t)F &\geq \frac{2Kt-1}{t}F - 2\langle \nabla F, \nabla f \rangle - \frac{t}{2} \left[\frac{(\text{Tr} h)^2}{m-n} + |h|^2 \right] + \frac{2F^2}{mt} \\
&\quad + \frac{4e^{-4Kt}(1 - e^{-2Kt})^2|\nabla f|^2F}{m} + \frac{2te^{-4Kt}(1 - e^{-2Kt})^2|\nabla f|^4}{m} + 2t\alpha_K(t)|\nabla f|^2 \\
&\quad + te^{-2Kt} \left(\frac{2\text{Tr} h}{m-n} \langle \nabla \phi, \nabla f \rangle + 2h(\nabla \phi, \nabla f) - \langle 2\text{div} h - \nabla \text{Tr}_g h + \nabla \partial_t \phi, \nabla f \rangle \right). \quad (60)
\end{aligned}$$

Set

$$S(\cdot) = \left\langle \frac{2\text{Tr} h}{m-n} \nabla \phi - 2\text{div} h - \nabla \text{Tr}_g h + \nabla \partial_t \phi, \cdot \right\rangle + 2h(\nabla \phi, \cdot), \quad (61)$$

and

$$A^2 = \max \left[|h|^2 + \frac{(\text{Tr} h)^2}{m-n} \right], \quad B = \max |S|.$$

Then

$$\begin{aligned}
(L - \partial_t)F &\geq \frac{2Kt-1}{t}F - 2\langle \nabla F, \nabla f \rangle - \frac{tA^2}{2} + \frac{2F^2}{mt} \\
&\quad + \frac{2te^{-4Kt}(1 - e^{-2Kt})^2|\nabla f|^4}{m} + 2t\alpha_K(t)|\nabla f|^2 - te^{-2Kt}B|\nabla f|.
\end{aligned}$$

Using the inequality

$$ax^4 + bx^2 + cx \geq -\frac{(b-\gamma)^2}{4a} - \frac{c^2}{4\gamma}, \quad (62)$$

where $\gamma > 0$ is any positive constant, we can derive that

$$\begin{aligned} & \frac{2te^{-4Kt}(1-e^{-2Kt})^2|\nabla f|^4}{m} + 2t\alpha_K(t)|\nabla f|^2 - te^{-2Kt}B|\nabla f| \\ & \geq -\frac{mt(2\alpha_K(t)-\gamma)^2}{8e^{-4Kt}(1-e^{-2Kt})^2} - \frac{te^{-4Kt}B^2}{4\gamma}. \end{aligned}$$

Thus

$$\begin{aligned} (L - \partial_t)F & \geq \frac{2Kt-1}{t}F - 2\langle \nabla F, \nabla f \rangle - \frac{tA^2}{2} + \frac{2F^2}{mt} \\ & \quad - \frac{mt(2\alpha_K(t)-\gamma)^2}{8e^{-4Kt}(1-e^{-2Kt})^2} - \frac{te^{-4Kt}B^2}{4\gamma}. \end{aligned} \quad (63)$$

Suppose at $(x_0, t_0) \in M \times [0, T]$, F achieves its maximum. Then $F(x_0, t_0) \geq 0$, and at (x_0, t_0) we have

$$\partial_t F \geq 0, \quad \Delta F \leq 0, \quad \nabla F = 0.$$

Thus

$$(L - \partial_t)F \leq 0.$$

Multiplying t_0 on the both sides of the last inequality, we have

$$0 \geq (2Kt_0 - 1)F + \frac{2F^2}{m} - \frac{t_0^2 A^2}{2} - \frac{mt_0^2(2\alpha_K(t_0) - \gamma)^2}{8e^{-4Kt_0}(1 - e^{-2Kt_0})^2} - \frac{t_0^2 e^{-4Kt_0} B^2}{4\gamma}, \quad (64)$$

Thus we obtain the following Li-Yau-Hamilton Harnack inequality on super Ricci flow

$$\begin{aligned} F & \leq \frac{m}{4} \left[(1 - 2Kt_0) + \sqrt{(1 - 2Kt_0)^2 + \frac{8t_0^2}{m} \left(\frac{A^2}{2} + \frac{m(2\alpha_K(t_0) - \gamma)^2}{8e^{-4Kt_0}(1 - e^{-2Kt_0})^2} + \frac{e^{-4Kt_0} B^2}{4\gamma} \right)} \right] \\ & \leq \frac{m}{2} \left[(1 - 2Kt_0)^+ + \sqrt{\frac{2}{m} \left(\frac{T^2 A^2}{2} + \frac{mt_0^2(2\alpha_K(t_0) - \gamma)^2}{8e^{-4Kt_0}(1 - e^{-2Kt_0})^2} + \frac{t_0^2 e^{-4Kt_0} B^2}{4\gamma} \right)} \right] \\ & \leq \frac{m}{2} \left[1 + \sqrt{\frac{A^2 T^2}{m} + \max_{t \in [0, T]} \frac{t^2(2\alpha_K(t) - \gamma)^2}{4e^{-4Kt}(1 - e^{-2Kt})^2} + \max_{t \in [0, T]} \frac{t^2 e^{-4Kt} B^2}{2m\gamma}} \right]. \end{aligned} \quad (65)$$

Note that when $B = 0$, we can take $\gamma = 0$ in (62) and in (65), i.e.,

$$F \leq \frac{m}{2} \left[1 + \sqrt{\frac{A^2 T^2}{m} + \max_{t \in [0, T]} \frac{t^2 \alpha_K^2(t) e^{4Kt}}{(1 - e^{-2Kt})^2}} \right], \quad (66)$$

and if $\alpha_K(t) = 0$, i.e., if

$$e^{-4Kt}(h + Ric_{m,n}(L) + K) - e^{-2Kt}h \geq 0, \quad (67)$$

we have

$$F \leq \frac{m}{2} \left[1 + \frac{TA}{\sqrt{m}} \right]. \quad (68)$$

In particular, when $A = B = 0$, and $Ric_{m,n}(L) \geq -K$, we recapture Hamilton's Harnack inequality [10]

$$F \leq \frac{m}{2}. \quad (69)$$

4.4 Hamilton's second order estimates for time dependent Witten Laplacian

Let $(M, g(t), \phi(t), t \in [0, T])$ be a compact Riemannian manifold equipped with a family of time dependent metrics $g(t)$ and potentials $\phi(t)$, $t \in [0, T]$. Let

$$\partial_t g = 2h.$$

Let u be a positive solution to the heat equation $\partial_t u = Lu$ associated with the time dependent Witten Laplacian $L = \Delta_{g(t)} - \nabla \phi(t) \cdot \nabla$. Let $P := (\partial_t - L - 2\nabla \log u \cdot \nabla)$, and let $\psi : [0, T] \rightarrow [0, \infty)$ be a C^1 -function. Set

$$F(x, t) = \psi(t) \left(\frac{Lu}{u} + \frac{|\nabla u|^2}{u^2} \right) - \left(m + 4 \log \left(\frac{A}{u} \right) \right).$$

By the Bochner formula, we have

$$\begin{aligned} PL \log u &= \partial_t L_t \log u + L(\partial_t - L) \log u - 2\nabla \log u \cdot \nabla L \log u \\ &= \partial_t L_t \log u + L|\nabla \log u|^2 - 2\nabla \log u \cdot \nabla L \log u \\ &= \partial_t L_t \log u + 2|\nabla^2 \log u|^2 + 2Ric(L)(\nabla \log u, \nabla u). \end{aligned}$$

On the other hand

$$\begin{aligned} P|\nabla \log u|^2 &= -\partial_t g(\nabla \log u, \nabla \log u) + 2\nabla \log u \cdot \nabla \partial_t \log u - L|\nabla \log u|^2 - 2\nabla \log u \cdot \nabla |\nabla \log u|^2 \\ &= -\partial_t g(\nabla \log u, \nabla \log u) + 2\nabla \log u \cdot \nabla L \log u - L|\nabla \log u|^2 \\ &= -(\partial_t g + 2Ric(L))(\nabla \log u, \nabla \log u) - 2|\nabla^2 \log u|^2. \end{aligned}$$

Combining the above two formulas together we have

$$\begin{aligned} PF &= \psi(t)P \left(\frac{Lu}{u} + \frac{|\nabla u|^2}{u^2} \right) + \psi'(t) \left(\frac{Lu}{u} + \frac{|\nabla u|^2}{u^2} \right) - 4P \log \left(\frac{A}{u} \right) \\ &= \psi(t) (\partial_t L_t \log u - 2|\nabla^2 \log u|^2 - 2(\partial_t g + Ric(L))(\nabla \log u, \nabla \log u)) \\ &\quad + \psi'(t)(L \log u + 2|\nabla \log u|^2) - 4|\nabla \log u|^2. \end{aligned}$$

By Lemma 4.1, we have

$$\partial_t L_t \log u = -\langle \partial_t g, \nabla^2 \log u \rangle - S_1(\nabla \log u),$$

where

$$S_1(\nabla \log u) = \langle \operatorname{div} \partial_t g - \frac{1}{2} \nabla \operatorname{Tr} \partial_t g - \nabla \partial_t \phi, \nabla \log u \rangle + \partial_t g(\nabla \phi, \nabla \log u).$$

Thus

$$\begin{aligned} PF &= -2\psi(|\nabla^2 \log u|^2 + \langle h, \nabla^2 \log u \rangle) - 2\psi(\partial_t g + Ric(L))(\nabla \log u, \nabla \log u) \\ &\quad + \psi' L \log u + 2(\psi' - 2)|\nabla \log u|^2 - \psi S_1(\nabla \log u) \\ &= -2\psi \left[|\nabla^2 \log u + \frac{h}{2}|^2 - \frac{|h|^2}{4} \right] + \psi' L \log u + 2(\psi' - (\partial_t g + Ric(L))\psi - 2)|\nabla \log u|^2 - \psi S_1(\nabla \log u) \\ &\leq -\frac{2\psi}{n} |\Delta \log u + \frac{\operatorname{Tr} h}{2}|^2 + \frac{\psi|h|^2}{2} + \psi' L \log u + 2(\psi' - (\partial_t g + Ric(L))\psi - 2)|\nabla \log u|^2 - \psi S_1(\nabla \log u). \end{aligned}$$

Using the inequality $(a+b)^2 \geq \frac{a^2}{1+\varepsilon} - \frac{b^2}{\varepsilon}$, and taking $\varepsilon = \frac{m-n}{n}$ for any $m \geq n$, we have

$$\frac{1}{n} |\Delta \log u + \frac{\operatorname{Tr} h}{2}|^2 \geq \frac{|L \log u|^2}{m} - \frac{|\nabla \phi \cdot \nabla \log u + \frac{\operatorname{Tr} h}{2}|^2}{m-n}.$$

Therefore

$$\begin{aligned}
PF &\leq -\frac{2\psi}{m}|L \log u|^2 + \psi' L \log u + \frac{2\psi|\nabla \phi \cdot \nabla \log u + \frac{\text{Tr} h}{2}|^2}{m-n} + \frac{\psi|h|^2}{2} \\
&\quad + 2(\psi' + \text{Ric}(L)\psi - 2)|\nabla \log u|^2 - \psi S_1(\nabla \log u) \\
&= -\frac{2\psi}{m}|L \log u|^2 + \psi' L \log u + \frac{\psi}{2} \left[\frac{|\text{Tr} h|^2}{m-n} + |h|^2 \right] \\
&\quad + 2(\psi' - (\partial_t g + \text{Ric}_{m,n}(L))\psi - 2)|\nabla \log u|^2 - \psi S(\nabla \log u),
\end{aligned}$$

where

$$S(\nabla \log u) = S_1(\nabla \log u) - \frac{2\text{Tr} h}{m-n} \nabla \phi \cdot \nabla \log u.$$

Suppose that

$$\partial_t g + \text{Ric}_{m,n}(L) \geq -Kg.$$

and denote $B = \max\{|S(v)| : v \in T.M, |v| = 1\}$. We have

$$\begin{aligned}
PF &\leq -\frac{2\psi}{m}|L \log u|^2 + \psi' L \log u + \frac{\psi}{2} \left[\frac{|\text{Tr} h|^2}{m-n} + |h|^2 \right] \\
&\quad + 2(\psi' + K\psi - 2)|\nabla \log u|^2 + 2\alpha|\nabla \log u|^2 + \frac{\psi^2 B^2}{8\alpha},
\end{aligned}$$

where α is any constant with $\alpha \in (0, 1)$. Taking $\psi(t) = (1 - \alpha)^{\frac{1-e^{-Kt}}{K}}$, then

$$\begin{aligned}
PF &\leq -\frac{2\psi}{m}|L \log u|^2 + \psi' L \log u - 2|\nabla \log u|^2 + \frac{\psi^2 B^2}{8\alpha} + \frac{\psi}{2} \left[\frac{|\text{Tr} h|^2}{m-n} + |h|^2 \right] \\
&\leq -\frac{2\psi}{m}|L \log u|^2 + \psi' L \log u - 2|\nabla \log u|^2 + C,
\end{aligned}$$

where

$$C = \frac{(1-\alpha)^2 B^2}{8\alpha K^2} + \frac{1-\alpha}{2K} \left[\frac{|\text{Tr} h|^2}{m-n} + |h|^2 \right].$$

Let $Q = -\frac{2\psi}{m}|L \log u|^2 + \psi' L \log u - 2|\nabla \log u|^2 + C$. Whenever $F \geq 0$, we have

$$\psi(L \log u + 2|\nabla \log u|^2) \geq m + 4 \log \left(\frac{A}{u} \right) \geq m,$$

which yields either $\psi L \log u \geq \frac{m}{2}$ or $2\psi|\nabla \log u|^2 \geq \frac{m}{2}$. In the case $\psi L \log u \geq \frac{m}{2}$, we have

$$Q \leq (\psi' - 1)L \log u - 2|\nabla \log u|^2 + C \leq C,$$

and in the case $2\psi|\nabla \log u|^2 \geq \frac{m}{2}$, we have

$$\begin{aligned}
Q &\leq -\frac{2\psi}{m}|L \log u|^2 + \psi' L \log u - \frac{m}{2\psi} + C \\
&\leq \frac{m}{2\psi} \left(\frac{\psi'^2}{4} - 1 \right) + C \\
&\leq C.
\end{aligned}$$

Thus, whenever $F \geq 0$, we have

$$PF \leq Q \leq C.$$

This yields that at any point where $F \geq 0$ we have

$$P(F - Ct) \leq 0.$$

Note that $F \leq 0$ at time $t = 0$. By the maximum principle, we can derive that

$$F \leq Ct, \quad \forall t \in [0, T].$$

Therefore we have proved the following Hamilton second order estimate for positive solutions to the heat equation $\partial_t u = Lu$ on compact Riemannian manifolds equipped with (a variant of) the (K, m) -super Ricci flow.

Theorem 4.5 *Let M be a compact Riemannian manifold with a family of Riemannian metrics $(g(t))$ and potentials $\phi(t)$, $t \in [0, T]$. Suppose that for some constants $m > n$ and $K \in \mathbb{R}$,*

$$\partial_t g + Ric_{m,n}(L) \geq -Kg, \quad \forall t \in [0, T].$$

Let u be a positive solution to the heat equation $\partial_t u = Lu$. Let $A = \max\{u(x, t) : x \in M, t \in [0, T]\}$. Then for any $\alpha \in (0, 1)$, we have

$$\frac{Lu}{u} + \frac{|\nabla u|^2}{u^2} \leq \frac{K}{(1-\alpha)(1-e^{-Kt})} \left[\frac{m}{2} + 4 \log \left(\frac{A}{u} \right) + Ct \right],$$

where

$$C = \frac{(1-\alpha)^2 B^2}{8\alpha K^2} + \frac{1-\alpha}{2K} \max \left[\frac{|\text{Tr} h|^2}{m-n} + |h|^2 \right],$$

with $B = \max\{|S(v)| : v \in T_x M, |v| = 1, x \in M, t \in [0, T]\}$, where

$$S(v) = \left\langle \frac{2\text{Tr} h}{m-n} \nabla \phi - 2\text{div} h + \nabla \text{Tr}_g h - \nabla \partial_t \phi, v \right\rangle + 2h(\nabla \phi, v).$$

In the case where $B = 0$, we can take $\alpha = 0$, i.e.,

$$\frac{Lu}{u} + \frac{|\nabla u|^2}{u^2} \leq \frac{K}{1-e^{-Kt}} \left[\frac{m}{2} + 4 \log \left(\frac{A}{u} \right) + Ct \right],$$

where

$$C = \frac{1}{2K} \max \left[\frac{|\text{Tr} h|^2}{m-n} + |h|^2 \right].$$

In the case where g and ϕ are time independent, and $Ric_{m,n}(L) \geq -Kg$, we have

$$\frac{Lu}{u} + \frac{|\nabla u|^2}{u^2} \leq \frac{K}{1-e^{-Kt}} \left[\frac{m}{2} + 4 \log \left(\frac{A}{u} \right) \right].$$

In particular, when $L = \Delta$ and $\phi(t) \equiv 0$, we have the following

Theorem 4.6 *Let M be a compact Riemannian manifold with a family of Riemannian metrics $(g(t), t \in [0, T])$. Suppose that for some constant $K \in \mathbb{R}$,*

$$\partial_t g + \text{Ric} \geq -Kg, \quad \forall t \in [0, T].$$

Let u be a positive solution to the heat equation $\partial_t u = \Delta u$. Let $A = \max\{u(x, t) : x \in M, t \in [0, T]\}$. Then for any $\alpha \in (0, 1)$ and any $m > n$, we have

$$\frac{\Delta u}{u} + \frac{|\nabla u|^2}{u^2} \leq \frac{K}{(1-\alpha)(1-e^{-Kt})} \left[\frac{m}{2} + 4 \log \left(\frac{A}{u} \right) + Ct \right],$$

where

$$C = \frac{(1-\alpha)^2 B^2}{8\alpha K^2} + \frac{1-\alpha}{2K} \max \left[\frac{|\text{Tr} h|^2}{m-n} + |h|^2 \right],$$

with $B = \max\{|S(v)| : v \in T_x M, |v| = 1, x \in M, t \in [0, T]\}$ for $S(v) = \langle \text{div} \partial_t g - \frac{1}{2} \nabla \text{Tr} \partial_t g, v \rangle$. In the case where g is the Ricci flow, i.e., $\partial_t g = -\text{Ric}$, we have $B = 0$, and we can take $\alpha = 0$, hence

$$\frac{\Delta u}{u} + \frac{|\nabla u|^2}{u^2} \leq \frac{K}{1-e^{-Kt}} \left[\frac{m}{2} + 4 \log \left(\frac{A}{u} \right) + Ct \right],$$

where

$$C = \frac{1}{2K} \max \left[\frac{|\text{Tr} h|^2}{m-n} + |h|^2 \right].$$

In the case where g and ϕ are time independent, and $\text{Ric} \geq -Kg$, we can take $m = n$ and we have Hamilton's second order estimate

$$\frac{\Delta u}{u} + \frac{|\nabla u|^2}{u^2} \leq \frac{K}{1-e^{-Kt}} \left[\frac{n}{2} + 4 \log \left(\frac{A}{u} \right) \right].$$

5 The Li-Yau and the Li-Yau-Hamilton Harnack inequalities on complete super Ricci flow

In this section we prove the Li-Yau Harnack inequality and the Li-Yau-Hamilton Harnack inequality on complete Riemannian manifolds equipped with variants of the (K, m) -super Ricci flow. In the literature, the Li-Yau Harnack inequality for heat equation $\partial_t u = \Delta u$ on complete Ricci flow has been studied by many authors. See [4, 6, 27] and references therein.

Similarly to [13], let η be a C^2 -function on $[0, \infty)$ such that $\eta = 1$ on $[0, 1]$ and $\eta = 0$ on $[2, \infty)$, with $-C_1 \eta^{1/2}(r) \leq \eta'(r) \leq 0$, and $\eta''(r) \geq C_2$, where $C, C_2 > 0$ are two constants. Let $\rho(x) = d(o, x)$ and define $\psi(x) = \eta(\rho(x)/R)$.

Let $Q_{2R, T} = \{(x, t) \in M \times [0, T] : d(x, x_0, t) \leq 2R, t \in [0, T]\}$. Let $\eta \in C^2([0, \infty), [0, 1])$ be such that $\eta(r) = 1$ on $[0, 1]$, $\eta = 0$ on $[2, \infty)$, $0 \leq \eta \leq 1$ on $[1, 2]$, $\eta'(r) \leq 0$, $\eta''(r) \geq -C$ and $|\eta'(r)|^2 \leq C\eta(r)$, where C is a positive constant. Define

$$\psi(x, t) = \psi(d(x, x_0, t)) = \eta \left(\frac{d(x, x_0, t)}{R} \right) = \eta \left(\frac{\rho(x, t)}{R} \right),$$

where $\rho(x, t) = d(x, x_0, t)$ denotes the geodesic distance between x and x_0 on $(M, g(t))$.

We need the following lemma.

Lemma 5.1 *Let M be a complete Riemannian manifold equipped with a family of time dependent metrics $g(t)$ and potentials $\phi(t)$, $t \in [0, T]$. Suppose that*

$$\partial_t g = 2h,$$

and for some function $\alpha_K : [0, T] \rightarrow \mathbb{R}$, it holds

$$e^{-4Kt}(h + \text{Ric}_{m,n}(L) + Kg) - e^{-2Kt}h \geq \alpha_K(t)g. \quad (70)$$

Suppose that $\text{Ric}_{m,n}(L) \geq -K_1$, $h \geq -K_2$, where K_1, K_2 are two positive constants. Then

$$(L - \partial_t)\psi \geq -C_1 K_2 \psi^{1/2} - \frac{C_1}{R}(m-1)\sqrt{K_1} \coth(\sqrt{K_1}\rho) - \frac{C_2}{R^2}.$$

Proof. By [13], under the condition $\text{Ric}_{m,n}(L) \geq -K_1$, on $(M, g(t), \phi(t))$, it holds

$$Ld(x_0, x, t) \leq (m-1)\sqrt{K_1}\rho \coth(\sqrt{K_1}\rho),$$

and

$$\begin{aligned} L\psi &= \eta'(d(x_0, x, t)/R) \frac{Ld(x_0, x, t)}{R} + \eta''(d(x_0, x, t)/R) \frac{|\nabla d(x_0, x, t)|^2}{R^2} \\ &\geq -\frac{C_1}{R}(m-1)\sqrt{K_1} \coth(\sqrt{K_1}\rho) - \frac{C_2}{R^2}. \end{aligned}$$

On the other hand, let $\gamma : [a, b] \rightarrow M$ be a fixed path such that $\gamma(a) = x$ and $\gamma(b) = y$. Let $S = \dot{\gamma}(s)$. Given a time $t_0 \in [0, T]$, assuming that γ is parameterize by the arc length with respect to metric $g(t_0)$ on M , then $|S| = 1$ at time $t = t_0$. Moreover, the evolution of the length of γ with respect to $g(t)$ is given by

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=t_0} L_{g(t)}(\gamma) &= \int_a^b \left. \frac{d}{dt} \right|_{t=t_0} \sqrt{g(t)(S, S)} ds \\ &= \frac{1}{2} \int_a^b \left. \frac{\partial_t g(t)(S, S)}{\sqrt{g(t)(S, S)}} \right|_{t=t_0} ds \\ &= \frac{1}{2} \int_a^b \left. \frac{\partial g(t)}{\partial t}(S, S) \right|_{t=t_0} ds. \end{aligned}$$

This yields, under the assumption $h \geq -K_2$, where $K_2 \geq 0$,

$$\partial_t d(x, y, t) = \int_a^b h(S, S) ds \geq -K_2 d(x, y, t).$$

Since $-C_1 \eta^{1/2}(r) \leq \eta'(r) \leq 0$, and $K_2 \geq 0$, it holds

$$\begin{aligned} -\partial_t \psi &= -\frac{\eta'(\rho/R) \partial_t d(x_0, x, t)}{R} \\ &\geq \frac{\eta'(\rho/R) K_2 d(x_0, x, t)}{R} \\ &\geq -\frac{C_1 K_2}{R} \psi^{1/2} d(x_0, x, t). \end{aligned}$$

Combining this with the lower bound of $L\psi$, we have

$$(L - \partial_t)\psi \geq -C_1 K_2 \psi^{1/2} - \frac{C_1}{R}(m-1)\sqrt{K_1} \coth(\sqrt{K_1}\rho) - \frac{C_2}{R^2}.$$

The proof of Lemma 5.1 is completed. \square

5.1 The Li-Yau Harnack inequality

In this subsection we prove the Li-Yau Harnack inequality for the positive solution to the heat equation $\partial_t u = L_t u$ of the time dependent Witten Laplacian on complete Riemannian manifolds equipped with a backward (α, K, m) -super Ricci flow.

Let u be a positive solution to the heat equation $\partial_t u = L_t u$. Let $f = \log u$. Then $(\partial_t - L)f = |\nabla f|^2$. For any $\alpha > 1$, let $F = t(|\nabla f|^2 - \alpha f_t)$.

Since ρ is Lipschitz on the complement of the cut locus of o , ψ is a Lipschitz function with support in $Q_{2R, T}$. As explained in Li and Yau [43], an argument of Calabi [3] allows us to apply the maximum principle to ψF . Let $(x_0, t_0) \in M \times [0, T]$ be a point where ψF achieves the maximum. Then, at (x_0, t_0) ,

$$\partial_t(\psi F) \geq 0, \quad \Delta(\psi F) \leq 0, \quad \nabla(\psi F) = 0,$$

which yields

$$(L - \partial_t)(\psi F) = \Delta(\psi F) - \nabla \phi \cdot \nabla(\psi F) - \partial_t(\psi F) \leq 0.$$

Note that

$$(L - \partial_t)(\psi F) = \psi(L - \partial_t)F + (L - \partial_t)\psi F + 2\nabla \psi \cdot \nabla F.$$

By Lemma 5.1, we have

$$(L - \partial_t)\psi \geq -C_1 K_2 \psi^{1/2} - \frac{C_1}{R}(m-1)\sqrt{K_1} \coth(\sqrt{K_1}\rho) - \frac{C_2}{R^2}.$$

Therefore, at (x_0, t_0) , we have

$$0 \geq \psi(L - \partial_t)F + 2\nabla \psi \cdot \nabla F - A(R, T)F, \quad (71)$$

where

$$A(R, T) := C_1 K_2 \psi^{1/2} + \frac{C_1}{R}(m-1)\sqrt{K_1} \coth(\sqrt{K_1}\rho) + \frac{C_2}{R^2}.$$

Denote

$$C_3 = C_3(m, K_1, K_2, R, T) = A(R, T) + 2|\nabla \psi|^2 \psi^{-1}.$$

We have

$$C_3(m, K_1, K_2, R, T) \leq C_1 K_2 + \frac{C}{R} + \frac{C_2}{R^2}.$$

Note that, at (x_0, t_0) , $\nabla \psi \cdot \nabla F = -\psi |\nabla \psi|^2 F$. Substituting (46) into (71), at (x_0, t_0) , we have

$$\begin{aligned} 0 &\geq \psi(L - \partial_t)F - A(R, T)F + 2\nabla \psi \cdot \nabla F \\ &\geq \psi(L - \partial_t)F - (A(R, T) + 2|\nabla \psi|^2 \psi^{-1})F \\ &\geq \frac{2\psi F^2}{m\alpha^2 t} - \left(\frac{\psi}{t} + C_3\right)F + \frac{4(\alpha-1)\psi |\nabla f|^2 F}{m\alpha^2} - \frac{2C_2}{R}\psi^{1/2}|\nabla f|F \\ &\quad + \psi t \left[\frac{2(\alpha-1)^2}{m\alpha^2} |\nabla f|^4 - 2K |\nabla f|^2 - \alpha B |\nabla f| - \frac{\alpha^2 A^2}{2} \right]. \end{aligned}$$

By the inequality $ax^2 - bx \geq \frac{4b^2}{a}$ and (49), and multiplying the both sides by ψt_0 , we have

$$\begin{aligned} 0 &\geq \frac{2(\psi F)^2}{m\alpha^2} - \left(\psi + C_3 t + \frac{m\alpha^2 C_2^2 t}{4(\alpha-1)R^2}\right)\psi F \\ &\quad - \psi^2 t^2 \left(\frac{m\alpha^2 (2K + \gamma)^2}{8(\alpha-1)^2} + \frac{\alpha^2 B^2}{4\gamma} + \frac{\alpha^2 A^2}{2} \right). \end{aligned}$$

This yields that, for any $(x, t) \in Q_{R,T}$,

$$\begin{aligned} F(x, t) &\leq (\psi F)(x_0, t_0) \\ &\leq \frac{m\alpha^2}{2} \left[1 + C_3 t_0 + \frac{m\alpha^2 C_2^2 t_0}{4(\alpha-1)R^2} + \sqrt{\frac{2}{m\alpha^2} \psi^2 t_0^2 \left(\frac{m\alpha^2 (2K+\gamma)^2}{8(\alpha-1)^2} + \frac{\alpha^2 B^2}{4\gamma} + \frac{\alpha^2 A^2}{2} \right)} \right] \\ &\leq \frac{m\alpha^2}{2} \left[1 + \left(C_1 K_2 + \frac{C}{R} + \frac{C_2}{R^2} + \frac{m\alpha^2 C_2^2}{4(\alpha-1)R^2} \right) T + T \sqrt{\frac{(2K+\gamma)^2}{4(\alpha-1)^2} + \frac{B^2}{2m\gamma} + \frac{A^2}{m}} \right]. \end{aligned}$$

Letting $R \rightarrow \infty$, we can derive, for all $\gamma > 0$, we have

$$F(x, t) \leq \frac{m\alpha^2}{2} \left[1 + \left(C_1 K_2 + \sqrt{\frac{(2K+\gamma)^2}{4(\alpha-1)^2} + \frac{B^2}{2m\gamma} + \frac{A^2}{m}} \right) T \right].$$

Therefore

$$\frac{|\nabla u|^2}{u} - \alpha \frac{\partial_t u}{u} \leq \frac{m\alpha^2}{2t} \left[1 + \left(C_1 K_2 + \sqrt{\frac{(2K+\gamma)^2}{4(\alpha-1)^2} + \frac{B^2}{2m\gamma} + \frac{A^2}{m}} \right) T \right].$$

5.2 The Li-Yau-Hamilton Harnack inequality

In this subsection we prove the Li-Yau-Hamilton Harnack inequality for the positive solution to the heat equation $\partial_t u = L_t u$ of the time dependent Witten Laplacian on complete Riemannian manifolds equipped with a variant of the (K, m) -super Ricci flow.

Let u be a positive solution to the heat equation $\partial_t u = L_t u$. Let $f = \log u$. Then $(\partial_t - L)f = |\nabla f|^2$. Let

$$F = te^{-4Kt} |\nabla f|^2 - te^{-2Kt} f_t.$$

Since ρ is Lipschitz on the complement of the cut locus of o , ψ is a Lipschitz function with support in $Q_{2R,T}$. As explained in Li and Yau [43], an argument of Calabi [3] allows us to apply the maximum principle to ψF . Let $(x_0, t_0) \in M \times [0, T]$ be a point where ψF achieves the maximum. Then, at (x_0, t_0) ,

$$\partial_t(\psi F) \geq 0, \quad \Delta(\psi F) \leq 0, \quad \nabla(\psi F) = 0,$$

which yields

$$(L - \partial_t)(\psi F) = \Delta(\psi F) - \nabla \phi \cdot \nabla(\psi F) - \partial_t(\psi F) \leq 0.$$

Note that

$$(L - \partial_t)(\psi F) = \psi(L - \partial_t)F + (L - \partial_t)\psi F + 2\nabla \psi \cdot \nabla F.$$

By Lemma 5.1, we have

$$(L - \partial_t)\psi \geq -C_1 K_2 \psi^{1/2} - \frac{C_1}{R} (m-1) \sqrt{K_1} \coth(\sqrt{K_1} \rho) - \frac{C_2}{R^2}.$$

Therefore, at (x_0, t_0) , we have

$$0 \geq \psi(L - \partial_t)F + 2\nabla \psi \cdot \nabla F - A(R, R)F,$$

where

$$A(R, T) := C_1 K_2 \psi^{1/2} + \frac{C_1}{R} (m-1) \sqrt{K_1} \coth(\sqrt{K_1} \rho) + \frac{C_2}{R^2}.$$

Denote

$$C(n, K, R, T) = A(R, T) + 2|\nabla\psi|^2\psi^{-1}.$$

We have

$$C(m, K_1, K_2, R, T) \leq C_1 K_2 + \frac{C}{R} + \frac{C_2}{R^2}.$$

Substituting (60) into (71), we have

$$\begin{aligned}
0 &\geq \psi(L - \partial_t)F - A(R, T)F + 2\nabla\psi \cdot \nabla F \\
&\geq \psi(L - \partial_t)F - (A(R, T) + 2|\nabla\psi|^2\psi^{-1})F \\
&\geq \psi \left[\frac{2[te^{-2Kt}(e^{-2Kt} - 1)|\nabla f|^2 - F]^2}{mt} - 2\langle \nabla F, \nabla f \rangle + \frac{(2Kt - 1)}{t}F \right] - C(m, K_1, K_2, R, T)F \\
&\quad + \psi t \left[-\frac{A^2}{2} + 2\alpha_K(t)|\nabla f|^2 - e^{-2Kt}B|\nabla f| \right] \\
&\geq \psi \left[\frac{2F^2}{mt} + \frac{2Kt - 1}{t}F - 2\langle \nabla F, \nabla f \rangle - \frac{tA^2}{2} + \frac{4e^{-2Kt}(1 - e^{-2Kt})|\nabla f|^2}{m}F \right] - C(m, K_1, K_2, R, T)F \\
&\quad + \psi t \left[\frac{2}{m}e^{-4Kt}(1 - e^{-2Kt})^2|\nabla f|^4 + 2\alpha_K(t)|\nabla f|^2 - e^{-2Kt}B|\nabla f| \right] \\
&\geq \frac{2\psi}{mt}F^2 + \frac{4e^{-2Kt}(1 - e^{-2Kt})|\nabla f|^2\psi}{m}F + 2F\langle \nabla\psi, \nabla f \rangle + \left[(2K - \frac{1}{t})\psi - C(m, K_1, K_2, R, T) \right] F \\
&\quad + \psi t \left[\frac{2}{m}e^{-4Kt}(1 - e^{-2Kt})^2|\nabla f|^4 + 2\alpha_K(t)|\nabla f|^2 - e^{-2Kt}B|\nabla f| - \frac{A^2}{2} \right] \\
&\geq \frac{2\psi}{mt}F^2 + \frac{4e^{-2Kt}(1 - e^{-2Kt})|\nabla f|^2\psi}{m}F - 2F|\nabla\psi||\nabla f| + \left[(2K - \frac{1}{t})\psi - C(m, K_1, K_2, R, T) \right] F \\
&\quad + \psi t \left[\frac{2}{m}e^{-4Kt}(1 - e^{-2Kt})^2|\nabla f|^4 + 2\alpha_K(t)|\nabla f|^2 - e^{-2Kt}B|\nabla f| - \frac{A^2}{2} \right] \\
&\geq \frac{2\psi}{mt}F^2 + \frac{4e^{-2Kt}(1 - e^{-2Kt})|\nabla f|^2\psi}{m}F - 2\frac{C_2}{R}F\psi^{1/2}|\nabla f| + \left[(2K - \frac{1}{t})\psi - C(m, K_1, K_2, R, T) \right] F \\
&\quad + \psi t \left[\frac{2}{m}e^{-4Kt}(1 - e^{-2Kt})^2|\nabla f|^4 + 2\alpha_K(t)|\nabla f|^2 - e^{-2Kt}B|\nabla f| - \frac{A^2}{2} \right].
\end{aligned}$$

Multiplying by t on both sides,

$$\begin{aligned}
0 &\geq \frac{2\psi}{m}F^2 + t\psi \frac{4e^{-2Kt}(1-e^{-2Kt})|\nabla f|^2}{m}F - 2t\frac{C_2}{R}F\psi^{1/2}|\nabla f| + [(2Kt-1)\psi - C(m, K_1, K_2, R, T)t]F \\
&\quad + \psi t^2 \left[\frac{2}{m}e^{-4Kt}(1-e^{-2Kt})^2|\nabla f|^4 + 2\alpha_K(t)|\nabla f|^2 - e^{-2Kt}B|\nabla f| - \frac{A^2}{2} \right] \\
&= \frac{2\psi}{m}F^2 + tF \left[\psi \frac{4e^{-2Kt}(1-e^{-2Kt})|\nabla f|^2}{m} - 2\frac{C_2}{R}\psi^{1/2}|\nabla f| \right] + [(2Kt-1)\psi - C(m, K_1, K_2, R, T)t]F \\
&\quad + \psi t^2 \left[\frac{2}{m}e^{-4Kt}(1-e^{-2Kt})^2|\nabla f|^4 + 2\alpha_K(t)|\nabla f|^2 - e^{-2Kt}B|\nabla f| - \frac{A^2}{2} \right] \\
&\geq \frac{2\psi}{m}F^2 + [(2Kt-1)\psi - C(m, K_1, K_2, R, T)t]F \\
&\quad + tF \left[\psi \frac{4e^{-2Kt}(1-e^{-2Kt})|\nabla f|^2}{m} - \frac{C_2m}{4e^{-2Kt}(1-e^{-2Kt})R^2} - \frac{4e^{-2Kt}(1-e^{-2Kt})}{m}\psi|\nabla f|^2 \right] \\
&\quad + \psi t^2 \left[\frac{2}{m}e^{-4Kt}(1-e^{-2Kt})^2|\nabla f|^4 + 2\alpha_K(t)|\nabla f|^2 - e^{-2Kt}B|\nabla f| - \frac{A^2}{2} \right] \\
&\geq \frac{2\psi}{m}F^2 + \left[(2Kt-1)\psi - C(m, K_1, K_2, R, T)t - \frac{C_2m}{4e^{-2Kt}(1-e^{-2Kt})R^2}t \right] F \\
&\quad - \psi t^2 \left[\frac{m(2\alpha_K(t) - \gamma)^2}{8e^{-4Kt}(1-e^{-2Kt})^2} + \frac{e^{-4Kt}B^2}{4\gamma} + \frac{A^2}{2} \right].
\end{aligned}$$

Notice that the above calculation is done at the point (x_0, t_0) . Since ψF reaches its maximum at this point, we can assume that $\psi F(x_0, t_0) > 0$. Thus

$$\begin{aligned}
0 &\geq \frac{2}{m}(\psi F)^2 - \left[1 + C(n, K, R, T)t + \frac{C_2}{4e^{-2Kt}(1-e^{-2Kt})R^2}t \right] (\psi F) \\
&\quad - \psi^2 t^2 \left[\frac{m(2\alpha_K(t) - \gamma)^2}{8e^{-4Kt}(1-e^{-2Kt})^2} + \frac{e^{-4Kt}B^2}{4\gamma} + \frac{A^2}{2} \right].
\end{aligned}$$

This yields that, for any $(x, t) \in Q_{R,T}$,

$$\begin{aligned}
F(x, t) &\leq (\psi F)(x_0, t_0) \\
&\leq \frac{m}{2} \left[1 + C(m, K_1, K_2, R, T)t_0 + \frac{C_2mt_0}{4e^{-2Kt_0}(1-e^{-2Kt_0})R^2} \right] \\
&\quad + \frac{\sqrt{m}}{2} \sqrt{A^2t_0^2 + \frac{m(2\alpha_K(t_0) - \gamma)^2t_0^2}{4e^{-4Kt_0}(1-e^{-2Kt_0})^2} + \frac{t_0^2e^{-4Kt_0}B^2}{2\gamma}} \\
&\leq \frac{m}{2} \left[1 + C(m, K_1, K_2, R, T)T + \frac{C_2mT}{4e^{-2KT}(1-e^{-2KT})R^2} \right] \\
&\quad + \frac{\sqrt{m}}{2} \sqrt{A^2T^2 + \max_{t \in [0, T]} \left(\frac{m(2\alpha_K(t) - \gamma)^2t^2}{4e^{-4Kt}(1-e^{-2Kt})^2} + \frac{t^2e^{-4Kt}B^2}{2\gamma} \right)}.
\end{aligned}$$

Let $R \rightarrow \infty$, we obtain

$$F \leq \frac{m}{2} \left[1 + C_1K_2T + \max_{t \in [0, T]} \frac{|2\alpha_K(t) - \gamma|t}{2e^{-2Kt}(1-e^{-2Kt})} \right] + \frac{\sqrt{m}}{2} \left(A + \frac{B}{\sqrt{2\gamma}} \right) T.$$

In the case $\alpha_K(t) = 0$, i.e., $e^{-4Kt}(h + Ric_{m,n}(L) + K) - e^{-2Kt}h \geq 0$, we have

$$F \leq \frac{m}{2} \left[1 + C_1K_2T + \max_{t \in [0, T]} \frac{\gamma t}{2e^{-2Kt}(1-e^{-2Kt})} \right] + \frac{\sqrt{m}}{2} \left(A + \frac{B}{\sqrt{2\gamma}} \right) T.$$

In addition, when $B = 0$, we can take $\gamma \rightarrow 0$, and we have

$$F \leq \frac{m}{2} \left[1 + C_1 K_2 T + \frac{AT}{\sqrt{m}} \right].$$

We can also extend Hamilton's second order estimate (i.e., Theorem 2.2, Theorem 4.5 and Theorem 4.6) to positive solutions to the heat equation associated with time dependent Witten Laplacian on complete Riemannian manifolds with variant of the (K, m) -super Ricci flow. To save the length of the paper, we omit it here.

Acknowledgement. The authors would like to thank D. Bakry, J.-M. Bismut, D. Elworthy, M. Ledoux, N. Mok, K.-T. Sturm, A. Thalmaier and F.-Y. Wang for helpful discussions and warm encouragements during the past years, and Dr. Y.-Z. Wang for careful preview and useful comments on the earlier versions of this paper. Part of this work has been done when the second author visited l'Institut des Hautes Etudes Scientifiques and l'Institut des Mathématiques de Toulouse de l'Université Paul Sabatier during November-December 2014, and l'Université Paris XIII during January-February 2016. He would like to thank Prof. D. Bakry, Prof. J.-M. Bismut and Prof. F. Nier for making this visit possible, and to thank IHES, UPS and Univ. Paris 13 for providing very nice environment to finish this work.

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